Asymptotic Expansions

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The interest in asymptotic analysis originated from the necessity of searching for approximations to functions close the point(s) of interest. Suppose we have a function \( f(x) \) of single real parameter \( x \) and we are interested in an approximation to \( f(x) \) for \( x \) “close to” \( x_0 \). The function \( f(x) \) may be undefined at \( x = x_0 \), however the limit \( \lim_{x \to x_0} f(x) = A \) must exists, finite, 0 or \( \infty \), as \( x_0 \) is approached from below or from above.

An approximation to \( f(x) \) for \( x \) close to \( x_0 \) is a function \( g(x) \) which is close to \( f(x) \) as \( x \to x_0 \). The formal definition is:

**Definition.** Let \( D \subseteq \mathbb{R}, x_0 \) a condensation point of \( D \), and \( f(x), g(x) \) and \( \eta(x) \) real functions defined on \( D \setminus x_0 \), i.e., on \( D \) excluding \( x_0 \). The function \( g(x) \) is an approximation to the function \( f(x) \) to order \( \eta(x) \) as \( x \to x_0 \) in \( D \) if, \(^1\)

\[
\lim_{x \to x_0} \frac{f(x) - g(x)}{\eta(x)} = 0,
\]

i.e., if for any small positive number \( \delta \) there exists a punctured neighborhood \( U_\delta(x_0) \) of \( x_0 \) such that

\[
|f(x) - g(x)| \leq \delta |\eta(x)| \quad \text{for} \quad x \in U_\delta(x_0) \cap D.
\]

The function \( \eta(x) \) is called gauge function and intuitively measures how close \( g(x) \) is to \( f(x) \) as \( x \to x_0 \).

Searching for an approximation to \( f(x) \) as \( x \to x_0 \) actually requires to know the rate at which \( f(x) \to A \) as \( x \to x_0 \), i.e., how the function \( f(x) \) approaches asymptotically the value \( A \) as \( x \to x_0 \) along the chosen direction. In other words, the asymptotic behavior of \( f(x) \) as \( x \to x_0 \).

The above example is a particular case of a single real variable function. In many problems we deal with functions \( f(x_1, \ldots, x_n; \epsilon_1, \ldots, \epsilon_m) \) depending on two sets of arguments: the set \((x_1, \ldots, x_n)\) called variables and the set \((\epsilon_1, \ldots, \epsilon_m)\) called parameters. The distinction between the two sets is not intrinsic and depends on the context.

For functions of more that one argument the above definition must be slightly modified. Suppose we look for an approximation to the function \( f(x; \epsilon) \) for \( x \)

\(^1\)The limit is evaluated in the domain \( D \setminus x_0 \). This will be always understood in the following.
in some region $D$ as the parameter $\epsilon$ approaches the value $\epsilon \to 0^+$. Without loss of generality we assume the distinguished value of $0^+$ for parameters. An approximation to $f(x; \epsilon)$ is a function $g(x; \epsilon)$ which is uniformly close to $f(x; \epsilon)$ as $\epsilon \to 0^+$ for all $x$ in $D$. Thus the above definition becomes:

**Definition.** Given the real functions $f(x, \epsilon)$ and $g(x, \epsilon)$ defined for $x$ in the domain $D \subseteq \mathbb{R}$ and $\epsilon$ in the interval $I : 0 < \epsilon \leq \epsilon_0$, and the real function $\eta(\epsilon)$ defined for $\epsilon \in I$. The function $g(x; \epsilon)$ is an approximation to the function $f(x; \epsilon)$ to order $\eta(\epsilon)$ for all $x \in D$ as $\epsilon \to 0^+$ if

$$\lim_{\epsilon \to 0^+} \frac{f(x; \epsilon) - g(x; \epsilon)}{\eta(\epsilon)} = 0 \quad \text{uniformly in } D,$$

i.e., if for any small positive number $\delta$ there exists a punctured neighborhood $I_\delta : 0 < \epsilon \leq \epsilon_\delta$ of $\epsilon = 0$ such that for all $x \in D$

$$|f(x; \epsilon) - g(x; \epsilon)| \leq \delta |\eta(\epsilon)| \quad \text{for } \epsilon \in I_\delta \cap I,$$

with $\delta$ and $\epsilon_\delta$ independent of $x$.

These definitions not only enlighten that searching for approximation close to points of interest naturally leads to an asymptotic analysis. What is most important, they show that the asymptotic behavior of functions can be studied by comparing them with known gauge functions. The simplest gauge functions are powers, for example for $\epsilon \to 0^+$

$$\ldots, \epsilon^{-2}, \epsilon^{-1}, 1, \epsilon, \epsilon^2, \ldots$$

but other choices are possible. The corollary is that fundamental to asymptotic analysis is to determine the relative order of functions.

1. **The Order Relations**

   The concept of order of magnitude of functions is central in asymptotic analysis. The precise mathematical definition of the intuitive idea of same order of magnitude, smaller order of magnitude and equivalent is provided by the Bachmann-Landau symbols “$O$”, “$o$” and “$\sim$”.

1.1. **The $O$ and $\sim$ symbols**

   The formal definition of the $O$-symbol used in asymptotical analysis is:

**Definition.** Let $D \subseteq \mathbb{R}$, $x_0$ a condensation point of $D$ and $f, g : D \setminus x_0 \to \mathbb{R}$ real functions defined on $D \setminus x_0$. Then $f = O(g)$ as $x \to x_0$ in $D$ if there exists a positive constant $C$ and a punctured neighborhood $U_C(x_0)$ of $x_0$ such that

$$|f(x)| \leq C |g(x)| \quad \text{for all } x \in U_C(x_0) \cap D.$$  \hspace{1cm} (1)
We say that \( f(x) \) is asymptotically bounded by \( g(x) \), or simply \( f(x) \) is “order big \( O \)” of \( g(x) \), as \( x \to x_0 \) in \( D \). If the ratio \( f/g \) is defined then (1) implies that \( |f/g| \) is bounded from above by \( C \) in \( U_C(x_0) \cap D \), or, equivalently that

\[
\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| < \infty.
\]

These definitions do not inquire into the fate of \( f \) and \( g \) in \( x_0 \); asymptotic analysis concerns with the asymptotic behavior of the functions in the neighborhood of the point \( x_0 \). In \( x_0 \) the functions \( f \) and \( g \) can be defined or not. If \( f = O(g) \) as \( x \to x_0 \) for all \( x_0 \) in \( D \) then \( f \) is bounded by \( g \) everywhere in \( D \). In this case we say that \( f = O(g) \) in \( D \).

The \( O \)-symbol provides a one-side bound, the statement \( f = O(g) \) as \( x \to x_0 \) indeed does not necessarily imply that \( f \) and \( g \) are of the same order of magnitude. Consider for example the functions \( f = x^\alpha \) and \( g = x^\beta \), where \( \alpha \) and \( \beta \) are constants with \( \alpha > \beta \). Then \( x^\alpha = O(x^\beta) \) as \( x \to 0^+ \) because the choice \( C = x^\alpha - x^\beta \) and \( U_C(0) : 0 < x < \delta \) satisfies the requirement (1). Clearly, the reverse \( x^\beta = O(x^\alpha) \) as \( x \to 0^+ \) is not true. Notice that \( x^\alpha/x^\beta \to 0 \) while \( x^\beta/x^\alpha \to +\infty \) as \( x \to 0^+ \).

Two functions \( f \) and \( g \) are strictly of the same order of magnitude if \( f = O(g) \) and \( g = O(f) \) as \( x \to x_0 \). To stress this point one sometimes introduces the notation

\[
f = O_s(g), \quad \text{as } x \to x_0
\]

for \( f \) strictly of order \( g \) to indicate \( f = O(g) \) and \( g = O(f) \) as \( x \to x_0 \). In this case \( \lim_{x \to x_0} |f/g| \) exists and it is neither zero or infinity. We shall not emphasize the distinction and use only the symbol “\( O \)”. However, the possible one-side nature of \( O \)-symbol should always be kept in mind.

If the \( \lim_{x \to x_0} (f/g) \) exists and is equal to \( 1 \) we say that \( f(x) \) is asymptotically equal to \( g(x) \) as \( x \to x_0 \), or simply \( f(x) \) “goes as” \( g(x) \) as \( x \to x_0 \), and write

\[
f \sim g, \quad \text{as } x \to x_0.
\]

For example, as \( x \to 0 \)

\[
\sin x \sim x, \quad \sin(7x) = O(x), \quad 1 - \cos x = O(x^2)
\]

\[
\log(1 + x) \sim x, \quad e^x \sim 1, \quad \sin(1/x) = O(1).
\]

The assertion \( 1 = O(\sin(1/x)) \) as \( x \to 0 \) is not true because \( \sin(1/x) \) vanishes in every neighborhood of \( x = 0 \). Similarly as \( x \to +\infty \)

\[
cosh(3x) = O(e^{3x}), \quad \log(1 + 7x) \sim \log x, \quad \sum_{n=1}^{N} a_n x^n = O(x^N).
\]

In the case of functions \( f(x; \epsilon) \), defined for \( x \) in a domain \( D \) and \( \epsilon \) in the interval \( I : 0 < \epsilon \leq \epsilon_0 \), we say that

\[
f(x; \epsilon) = O \left[ g(x; \epsilon) \right] \quad \text{as } \epsilon \to 0^+ \text{ in } D,
\]

3
if for each \( x \) in \( D \) there exist a positive number \( C_x \) and a punctured neighborhood \( I_\delta(x) : 0 < \epsilon \leq \epsilon_\delta(x) \) of \( \epsilon = 0 \), such that

\[
|f(x; \epsilon)| \leq C_x |g(x; \epsilon)| \quad \text{for all } \epsilon \in I_\delta(x) \cap I.
\]

(2)

If the ratio \( f/g \) exists then from (2) follows that

\[
\lim_{\epsilon \to 0^+} \frac{|f(x; \epsilon)|}{|g(x; \epsilon)|} < \infty, \quad \text{for all } x \in D.
\]

The limit may depend on \( x \). If the positive constant \( C_x \) and \( \epsilon_\delta(x) \) do not depend on \( x \) for all \( x \in D \) then \( f(x; \epsilon) = O(g(x; \epsilon)) \) as \( \epsilon \to 0^+ \) uniformly in \( D \). For example

\[
\cos(x + \epsilon) = O(1) \quad \text{as } \epsilon \to 0 \text{ uniformly in } \mathbb{R}, \]

\[
e^{\epsilon x} - 1 = O(\epsilon) \quad \text{as } \epsilon \to 0 \text{ nonuniformly in } \mathbb{R}.
\]

A relation not uniform in \( D \) can be uniform in a subdomain of \( D \). For example, the second example is uniform in the subdomain \([0 : 1] \subset \mathbb{R}\).

As in the case of a single variable function, if \( \lim_{\epsilon \to 0^+} (f/g) = 1 \) for all \( x \in D \) we say that \( f \) is asymptotic equal to \( g \) in \( D \) and write

\[
f(x; \epsilon) \sim g(x; \epsilon) \quad \text{as } \epsilon \to 0^+ \text{ for all } x \in D.
\]

1.2. The \( o \)-symbol

The formal definition of the \( o \)-symbol is:

**Definition.** Let \( D \subseteq \mathbb{R}, x_0 \) a condensation point of \( D \) and \( f, g : D \setminus x_0 \to \mathbb{R} \) real functions defined on \( D \setminus x_0 \). We say \( f = o(g) \) as \( x \to x_0 \) in \( D \) if for every \( \delta > 0 \) there exists a punctured neighborhood \( U_\delta(x_0) \) of \( x_0 \) such that

\[
|f(x)| \leq \delta |g(x)| \quad \text{for all } x \in U_\delta(x_0) \cap D.
\]

(3)

This inequality indicates that \( |f| \) becomes arbitrarily small compared to \( |g| \) as \( x \to x_0 \) in \( D \). Thus, if the ratio \( f/g \) is defined, (3) implies that

\[
\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = 0.
\]

Notice that \( f = o(g) \) as \( x \to x_0 \) always implies \( f = O(g) \) as \( x \to x_0 \) not in strict sense. The converse is not true. Asymptotic equivalence \( f(x) \sim g(x) \) as \( x \to x_0 \) means that \( f(x) = g(x) + o(g(x)) \) as \( x \to x_0 \).

If \( f = o(g) \) as \( x \to x_0 \) we say that \( f(x) \) is *asymptotically smaller* than \( g(x) \) as \( x \to x_0 \), or simply \( f(x) \) is "order little \( o \)" of \( g(x) \) as \( x \to x_0 \). Often the alternative notation \( f \ll g \) as \( x \to x_0 \), or \( f \) "much smaller than" \( g \) as \( x \to x_0 \), is used in place of \( f = o(g) \).
For example, as \( x \to 0 \)

\[
x^3 = o(x), \quad \sin(7x) = o(1), \\
\cos x = o(x^{-1}), \quad \log(1 + x^2) = o(x), \\
e^x - 1 = o(x^{-1/3}), \quad e^{-1/x^2} = o(x^n) \text{ for all } n \in \mathbb{N},
\]

while as \( x \to +\infty \)

\[
x^2 = o(x^3), \quad \log x = o(x), \quad \sum_{n=1}^{N} a_n x^n = o(x^{N+1}).
\]

Generalization to functions \( f(x, \epsilon) \), defined for \( x \) in a domain \( D \) and \( \epsilon \) in the interval \( I: 0 < \epsilon \leq \epsilon_0 \), is immediate. The statement

\[
f(x, \epsilon) = o\left[g(x, \epsilon)\right] \quad \text{as } \epsilon \to 0^+ \text{ for } x \in D,
\]

means that for each \( x \) in \( D \) and any given \( \delta > 0 \) there exist a punctured neighborhood \( I_\delta(x) : 0 < \epsilon \leq \epsilon_\delta(x) \) of \( \epsilon = 0 \), such that

\[
|f(x, \epsilon)| \leq \delta |g(x, \epsilon)| \quad \text{for all } \epsilon \text{ in } I_\delta(x) \cap I.
\]

This inequality implies that if the ratio \( f/g \) exists then

\[
\lim_{\epsilon \to 0^+} \left| \frac{f(x; \epsilon)}{g(x; \epsilon)} \right| = 0, \quad \text{for all } x \in D.
\]

If \( \epsilon_\delta \) is independent of \( x \) we say that \( f(x; \epsilon) = o(g(x; \epsilon)) \) as \( \epsilon \to 0^+ \) uniformly in \( D \). For example

\[
\cos(x + \epsilon) = o(\epsilon^{-1}) \quad \text{as } \epsilon \to 0^+ \text{ uniformly in } \mathbb{R}, \\
e^{\epsilon x} - 1 = o(\epsilon^{1/2}) \quad \text{as } \epsilon \to 0^+ \text{ nonuniformly in } \mathbb{R}, \\
e^{\epsilon x} - 1 = o(\epsilon^{1/2}) \quad \text{as } \epsilon \to 0^+ \text{ uniformly in } D: 0 \leq x \leq 1.
\]

The uniform validity of the order relation is more important for the \( \mathcal{O} \)-symbol. In asymptotic analysis the \( \mathcal{O} \) (in strict sense) and \( \sim \) symbols are usually more relevant than the \( o \)-symbol because the latter hide so much information. We are more interested to know how a function behaves close to some point, rather than just knowing that it is much smaller than some other functions. For example, \( \sin x = x + o(x^2) \) as \( x \to 0 \) tells us that \( \sin x - x \) vanishes faster than \( x^2 \) as \( x \to 0 \), while \( \sin x = x + O(x^3) \) as \( x \to 0 \) (in strict sense) tells us that \( \sin x - x \) vanishes as \( x^3 \) as \( x \to 0 \).

2. Asymptotic Expansion

An asymptotic expansion describes the asymptotic behavior of a function close to a point in terms of some reference functions, the gauge functions.
The definition was introduced by Poincaré (1886) and provides a mathematical framework for the use of many divergent series.

We shall call the set of functions \( \varphi_n : D \setminus x_0 \to \mathbb{R}, n = 0, 1, \ldots \), an asymptotic sequence as \( x \to x_0 \) in \( D \setminus x_0 \), if for each \( n \) we have

\[
\varphi_{n+1}(x) = o\left[\varphi_n(x)\right] \quad \text{as} \quad x \to x_0 \quad \text{in} \quad D.
\]

Some examples of asymptotic sequence are:

**Example 2.1.** The functions \( \varphi_n(x) = (x - x_0)^n \) form an asymptotic sequence as \( x \to x_0 \).

**Example 2.2.** The functions \( \varphi_n(x) = x^n/2, \varphi_n(x) = (\log x)^n \) and \( \varphi_n(x) = (\sin x)^n \) are examples of asymptotic sequences as \( x \to 0^+ \), while the functions \( \varphi_n(x) = x^{-n} \) form an asymptotic sequence as \( x \to +\infty \).

**Example 2.3.** The sequence \( \{\log x, x^2, x^3, \ldots\} \) form an asymptotic sequence as \( x \to 0^+ \).

**Example 2.4.** The sequence \( \{\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots\} \) form an asymptotic sequence as \( x \to 0^+ \), while the sequence \( \{\ldots, x^2, x, 1, x^{-1}, x^{-2}, \ldots\} \) form an asymptotic sequence as \( x \to +\infty \).

The asymptotic expansion of a function \( f(x) \) is defined as:

**Definition.** Let the set of functions \( \varphi_n(x), n = 0, 1, 2, \ldots \), defined on the domain \( D \setminus x_0 \) form an asymptotic sequence as \( x \to x_0 \) in \( D \) and \( a_n \) a sequence of numbers. The formal series \( \sum_{n=0}^{\infty} a_n \varphi_n(x) \) is an asymptotic expansion of \( f(x) \) as \( x \to x_0 \) in \( D \) with gauge functions \( \varphi_n(x) \) if, and only if, for any \( N \):

\[
f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o\left[\varphi_N(x)\right] \quad \text{as} \quad x \to x_0 \quad \text{in} \quad D,
\]

or, equivalently,

\[
f(x) - \sum_{n=0}^{N-1} a_n \varphi_n(x) = O\left[\varphi_N(x)\right] \quad \text{as} \quad x \to x_0 \quad \text{in} \quad D. \quad (4)
\]

In this case we shall write

\[
f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \quad \text{as} \quad x \to x_0 \quad \text{in} \quad D.
\]

Asymptotic expansion of this form are also called asymptotic series.

The definition of the asymptotic expansion of a single variable function is not general enough to handle functions of more than one variable. For them we need the more general definition:
Definition. Let the functions \( \varphi_n(\epsilon) \), \( n = 0, 1, 2, \ldots \), with \( \epsilon \in I : 0 < \epsilon \leq \epsilon_0 \) form an asymptotic sequence as \( \epsilon \to 0^+ \). We say that the sequence of functions \( g_n(x; \epsilon) \) form an uniform asymptotic sequence of approximations to \( f(x; \epsilon) \) in the domain \( x \in D \) with gauge functions \( \varphi_n(\epsilon) \) as \( \epsilon \to 0^+ \) if

\[
\lim_{\epsilon \to 0^+} \frac{f(x; \epsilon) - g_n(x; \epsilon)}{\varphi_n(\epsilon)} = 0, \quad \text{for all } n,
\]

uniformly in \( D \).

The boundary of the domain \( D \) where the approximation is uniformly valid may depend on \( \epsilon \), and also on \( n \). Domains with \( \epsilon \)-dependent boundaries are essential in asymptotic analysis because the gauge functions are not unique.

The definition (5) is not an asymptotic expansion of \( f(x; \epsilon) \); to find an asymptotic expansion of \( f(x; \epsilon) \) one constructs a sequence of functions \( a_n(x; \epsilon) \) such that

\[
g_n(x; \epsilon) = \sum_{m=0}^{n} a_m(x; \epsilon),
\]

and (5) holds. The formal series \( \sum_{n=0}^{\infty} a_n(x; \epsilon) \) is a generalized uniform asymptotic expansion of \( f(x; \epsilon) \) and we write

\[
f(x; \epsilon) \sim \sum_{n=0}^{\infty} a_n(x; \epsilon) \quad \text{as } \epsilon \to 0^+,
\]

uniformly in \( D \). The formal expression (7) is just the restatement of the definition (5) with the particular choice (6). The leading term \( a_0(x; \epsilon) \) of the series is called asymptotic representation of \( f(x; \epsilon) \) as \( \epsilon \to 0^+ \). Notice that nothing is said about the convergence of the series. The series (7) is just formal and generally it does not converge; if it converges it may not converge to \( f(x; \epsilon) \).

In most cases the functions \( a_n(x; \epsilon) \) do not have a simple expression in terms of \( \varphi_n(\epsilon) \), and finding them may be rather difficult. To overcome this problem, Poincaré introduced a definition in terms of the functions \( \varphi_n(\epsilon) \).

Definition. The uniform asymptotic expansion of the function \( f(x; \epsilon) \) in the domain \( x \in D \) and \( \epsilon \) in the interval \( I : 0 < \epsilon \leq \epsilon_0 \), as \( \epsilon \to 0^+ \) in the sense of Poincaré, or simply of Poincaré type, is an asymptotic expansion of the form

\[
f(x; \epsilon) \sim \sum_{n=0}^{\infty} a_n(x) \varphi_n(\epsilon), \quad \text{as } \epsilon \to 0^+ \text{ uniformly for } x \in D.
\]

The functions \( a_n(x) \) are independent of \( \epsilon \) and play the role of coefficients of the asymptotic expansion.

Poincaré type asymptotic expansions have useful properties not shared by generalized asymptotic expansion, for example they can be added or integrated term by term, hence they are largely used in asymptotic analysis. Their form is, however, not general enough to handle all problems that can be studied with
perturbation techniques. Nevertheless, they can generally be used as starting
point to construct more suitable generalized asymptotic expansions.

Since Poincaré type expansions simply extend the definition of asymptotic
expansion of functions $f(x)$ to functions $f(x; \epsilon)$, they are usually introduced
using the following equivalent definition:

**Definition.** Let the functions $\varphi_n(\epsilon), n = 0, 1, 2, \ldots$, with $\epsilon \in I : 0 < \epsilon \leq \epsilon_0$
form an asymptotic sequence as $\epsilon \to 0^+$ and $a_n(x)$ a sequence of functions independent of $\epsilon$ defined on the domain $D$. Then the formal series $\sum_{n=0}^{\infty} a_n(x) \varphi_n(\epsilon)$ is a uniform asymptotic expansion of Poincaré type of $f(x; \epsilon)$ in the domain $x \in D$ with gauge functions $\varphi_n(\epsilon)$ as $\epsilon \to 0^+$ if, and only if, for any $N$:

$$f(x; \epsilon) - \sum_{n=0}^{N} a_n(x) \varphi_n(\epsilon) = o[\varphi_N(\epsilon)], \quad \text{as } \epsilon \to 0^+,$$

or, equivalently,

$$f(x; \epsilon) - \sum_{n=0}^{N-1} a_n(x) \varphi_n(\epsilon) = O[\varphi_N(\epsilon)], \quad \text{as } \epsilon \to 0^+,$$

uniformly for $x \in D$.

2.1. Uniqueness of Poincaré type Expansions and Subdominance

The asymptotic expansion of a function is not unique because there exists
an infinite number of asymptotic sequences that can be used in the expansion.
However, given an asymptotic sequence of gauge functions $\varphi_n$ the Poincaré type
expansion in terms of $\varphi_n$ is unique. Consider for simplicity a function $f(x)$ of
a single variable, the result holds for the Poincaré type expansions of functions
$f(x; \epsilon)$ as well. From the definition (4) it follows that

$$\lim_{x \to x_0} \frac{f(x) - \sum_{n=0}^{N-1} a_n(x) \varphi_n(x)}{\varphi_N(x)} = a_N.$$

Thus the coefficients $a_n$ are uniquely determined by the gauge functions $\varphi_n$; if
the function $f(x)$ admits an asymptotic expansion with respect to the sequence
of gauge functions $\varphi_n(x)$ then the expansion is unique.

The converse is not true. For any asymptotic sequence $\varphi_n(x)$ as $x \to x_0$
there always exists a function $\psi(x)$ which is *transcendentally small* with respect
to the sequence, i.e.,

$$\psi = o(\varphi_n), \quad \forall n, \quad \text{as } x \to x_0.$$

One may then form a sequence of transcendentally small terms, by say positive
powers of $\psi$. As a consequence an asymptotic expansion does not determines
the function $f$ uniquely. Different functions may have the same asymptotic
expansion with respect to the given asymptotic sequence of gauge functions $\varphi_n$. 
Example 2.5. For any constant \( c \neq 0 \), the functions

\[ f(x) = \frac{1}{1-x} \quad \text{and} \quad g(x) = \frac{1}{1-x} + ce^{-1/x}, \]

have the same asymptotic expansion as \( x \to 0^+ \) in terms of the gauge functions \( \varphi_n(x) = x^n \):

\[ f(x) \sim g(x) \sim 1 + x + x^2 + \ldots \quad \text{as} \quad x \to 0^+, \]

because \( e^{-1/x} \) is transcendentally small with respect to \( x^n \): \( e^{-1/x} = o(x^n) \) as \( x \to 0^+ \) for every \( n \in \mathbb{N} \).

An asymptotic expansion establishes an equivalence relation among functions: if for all \( n \)

\[ f(x) - g(x) = o[\varphi_n(x)] \quad \text{as} \quad x \to x_0, \]

then the functions \( f \) and \( g \) are asymptotically equivalent as \( x \to x_0 \) with respect to the gauge functions \( \varphi_n \). The function \( h(x) = f(x) - g(x) \) is said subdominant to the asymptotic expansion with gauge functions \( \varphi_n \) as \( x \to x_0 \). An asymptotic expansion is, thus, asymptotic to a class of functions that differ one another by subdominant functions transcendentally small with respect to the gauge function \( \varphi_n \) as \( x \to x_0 \).

Notice that the concept of transcendentally smallness is an asymptotic one. Transcendentally small terms might be numerically important for values of \( x \) not too close to \( x_0 \), even for relatively small non-zero values of \( x - x_0 \).

2.2. Uniformly vs Nonuniformly valid Asymptotic Expansions

In perturbation calculations one often encounters situations where the quantity to be expanded is a function \( f(x; \epsilon) \), where \( \epsilon \) is a (small) perturbation parameter and \( x \) is a variable independent of \( \epsilon \) varying in a domain \( D \). In these cases one generally constructs an asymptotic expansion of the Poincaré type (8) with some gauge functions \( \varphi_n(\epsilon) \). The functions \( \varphi_n(\epsilon) \) are not uniquely determined by the problem. Rather, these are usually found in the course of the solution of the problem. Due to this arbitrariness different asymptotic approximations can be constructed. Only asymptotic expansions uniform in the domain \( D \), however, lead to useful approximations to \( f(x; \epsilon) \).

An asymptotic expansion of \( f(x; \epsilon) \), e.g. \( \sum_{n=0}^{\infty} a_n(x) \varphi_n(\epsilon) \), is only formal. However the partial sum of the first \( N \) terms is a well defined object for any finite \( N \) and gives an approximation to the function \( f(x; \epsilon) \). Thus, we can write

\[ f(x; \epsilon) - \sum_{n=0}^{N-1} a_n(x) \varphi_n(\epsilon) = R_N(x; \epsilon) \]

where \( R_N(x; \epsilon) \) is the reminder or error of the approximation. The asymptotic properties of the gauge functions \( \varphi_n(\epsilon) \) ensures that for any fixed \( N \) and \( x \in D \):

\[ R_N(x; \epsilon) = O[\varphi_N(\epsilon)], \quad \text{as} \quad \epsilon \to 0. \quad (9) \]
The asymptotic expansion gives both an approximation to \( f(x; \epsilon) \) as \( \epsilon \to 0 \) and the estimation of the error in the approximation. The error \( R_N(x; \epsilon) \) is in general a function of both \( \epsilon \) and \( x \). For a useful approximation, however, the error must not depend upon the value of \( x \), hence (9) must be valid uniformly in \( D \). Useful asymptotic expansions of \( f(x; \epsilon) \) are only those which are uniform in \( D \).

Notice that since \( \phi_n(\epsilon) = o[\phi_{n-1}(\epsilon)] \) as \( \epsilon \to 0 \) the requirement of uniform validity implies that each coefficient \( a_n(x) \) of a Poincaré type expansion cannot be more singular than \( a_{n-1}(x) \) for all \( x \) in the domain \( D \). In other words, as \( \epsilon \to 0 \) each term \( a_n(x)\phi_n(\epsilon) \) must be a small correction to the preceding term irrespective of the value of \( x \).

**Example 2.6.** The expansion

\[
\sin(x + \epsilon) = \left(1 - \frac{\epsilon^2}{2} + \ldots\right) \sin x + \left(\epsilon - \frac{\epsilon^3}{3!} + \ldots\right) \cos x
\]

is a uniform asymptotic expansion in \( \mathbb{R} \). The coefficients of the power \( \epsilon^n \) are bounded for all \( x \in \mathbb{R} \), thus \( a_n(x) \) is not more singular than \( a_{n-1}(x) \) and the expansion is uniform in \( \mathbb{R} \).

**Example 2.7.** Consider the function

\[
f(x; \epsilon) = \frac{1}{x + \epsilon},
\]

defined for all \( x \geq 0 \) and \( \epsilon > 0 \). If \( x > 0 \), then

\[
f(x; \epsilon) \sim \frac{1}{x} \left[1 - \frac{\epsilon}{x} + \frac{\epsilon^2}{x^2} + \ldots\right], \quad \text{as } \epsilon \to 0^+.
\] (10)

Each term of this expansion is singular at \( x = 0 \), and more singular than the preceding one. The expansion is not uniform in the domain \( D : x \geq 0 \). It is, nevertheless, uniform in the domain \( D_1 : 0 < x_0 \leq x \) where \( x_0 \) is a positive number independent of \( \epsilon \). Uniformity breaks down as we get too close to \( x = 0 \), signaling that some change must occur near \( x = 0 \). Indeed for \( x = 0 \) the expansion has a completely different form:

\[
f(0; \epsilon) \sim \frac{1}{\epsilon}, \quad \text{as } \epsilon \to 0^+.
\] (11)

Generally speaking, the uniform validity of the expansion (10) breaks when two successive terms become of the same order, i.e.,

\[
\frac{\epsilon}{x} = O(1) \quad \Rightarrow \quad x = O(\epsilon) \quad \text{as } \epsilon \to 0^+.
\]
If we introduce the variable \( \xi = x/\epsilon \) and rewrite \( f(x; \epsilon) \) in terms of \( \xi \) the different asymptotic expansion emerges:

\[
f(\epsilon \xi; \epsilon) \sim \frac{1}{\epsilon} \frac{1}{\xi + 1}, \quad \text{as} \quad \epsilon \to 0^+.
\] (12)

The expansion (12) is uniform in the domain \( D_2 : 0 \leq \xi \leq \xi_0 < 1/\epsilon \), with \( \xi_0 \) a constant independent of \( \epsilon \). Notice that \( D_2 \) differs from \( D_1 \). Nevertheless the expansion (12) matches with (11) in the limit \( \xi \to 0^+ \) and with (10) as \( \xi \gg 1 \).

We shall give the precise meaning of the vague word “matches” later, discussing asymptotic matching.

2.3. Asymptotic versus Convergent series

Convergence of a series is an intrinsic property of the coefficients of the series, one can indeed prove the convergence of a series without knowing the function to which it converges. Asymptotic is, instead, a relative concept. It concerns both the coefficients of the series and the function to which the series is asymptotic. To prove that an asymptotic series is asymptotic to the function \( f(x) \) both the coefficients of the series and the function \( f(x) \) must be considered. As a consequence, a convergent series need not be asymptotic and an asymptotic series need not be convergent.

To understand the difference between convergence and asymptotic consider the asymptotic series \( \sum_{n=0}^{\infty} a_n \varphi_n(x) \), where \( a_n \) are some coefficients and \( \varphi_n(x) \) an asymptotic sequence as \( x \to x_0 \) in a domain \( D \). The series is in general only formal because it may not converge. The partial sum

\[
S_N(x) = \sum_{n=0}^{N-1} a_n \varphi(x)
\]
is, nevertheless, well defined for any \( N \). The convergence of the series is concerned with the behavior of \( S_N(x) \) as \( N \to \infty \) for fixed \( x \), whereas the asymptotic property as \( x \to x_0 \) with the behavior of \( S_N(x) \) as \( x \to x_0 \) for fixed \( N \).

A convergent series defines for each \( x \) a unique limiting sum \( S_\infty(x) \). However, at difference with an asymptotic expansion, it does not provide information on how rapidly it converges nor on how well \( S_N(x) \) approximates \( S_\infty(x) \) for fixed \( N \). An asymptotic series, on the contrary, does not define a unique limiting function \( f(x) \), but rather a class of asymptotically equivalent functions. Hence it does not provide an arbitrary accurate approximation to \( f(x) \) for \( x \neq x_0 \). Still, the partial sums \( S_N(x) \) provide good approximations to the value of \( f(x) \) for \( x \) sufficiently close to \( x_0 \). The approximation, however, depends on how \( x \) is close to \( x_0 \). This leads to the problem of the optimal truncation of the asymptotic expansion, i.e., the optimal number of terms to be retained in the partial sum.\(^2\)

The following example illustrate these differences.

\(^2\)The optimal truncation can be affected by the presence of transcendentally small sub-dominant functions because they may give finite contributions.
Example 2.8. Consider the error function, or Gauss error function, defined as

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt. \]  

(13)

Since the Maclaurin series of \( e^z \) is absolutely convergent for all \( z \in \mathbb{C} \), expanding \( e^{-t^2} \) in power series of \( -t^2 \) and integrating term by term, we obtain the convergent power series expansion of \( \text{erf}(x) \):

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \left[ x - \frac{1}{3} x^3 + \frac{1}{2 \cdot 5} x^5 + \ldots \right] = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}. \]

The series converges for all \( x \in \mathbb{R} \) because the ratio between the \( (n+1) \)-th and the \( n \)-th term

\[ \left| \frac{u_{n+1}}{u_n} \right| = \frac{2n+1}{2n+3} \frac{x^2}{n+1} \]

is smaller than 1 for all \( x \in \mathbb{R} \) and \( n \) large enough. For moderate values of \( x \) the convergence is, however, very slow. For example, 31 terms are needed to have an approximation to \( \text{erf}(3) \) with an accuracy of \( 10^{-5} \). A better result can be obtained using an asymptotic expansion.

To derive an asymptotic expansion of \( \text{erf}(x) \) we rewrite (13) as,

\[ \text{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt \]

\[ = 1 - \frac{1}{\sqrt{\pi}} \int_x^\infty s^{-1/2} e^{-s} \, ds. \]

Integrating by parts twice we get:

\[ \text{erf}(x) = 1 - \frac{1}{\sqrt{\pi}} \left[ \frac{e^{-x^2}}{x} - \frac{1}{2} \int_{x^2}^\infty s^{-3/2} e^{-s} \, ds \right] \]

\[ = 1 - \frac{1}{\sqrt{\pi}} \left[ \frac{e^{-x^2}}{x} - \frac{e^{-x^2}}{2x^3} + \frac{3}{4} \int_{x^2}^\infty s^{-5/2} e^{-s} \, ds \right]. \]

The procedure can be easily iterated to any order \( N \). Introducing the functions

\[ F_n(x) = \int_{x^2}^\infty s^{-n-1/2} e^{-s} \, ds \]

\[ = - \left[ s^{-n-1/2} e^{-s} \right]_{x^2}^\infty + \left( n + \frac{1}{2} \right) \int_{x^2}^\infty s^{-n-1/2} e^{-s} \, ds \]

\[ = \frac{e^{-x^2}}{x^{2n+1}} - \left( n + \frac{1}{2} \right) F_{n+1}(x), \]

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the error function \( \text{erf}(x) \) can be written as

\[
\text{erf}(x) = 1 - \frac{1}{\sqrt{\pi}} F_0(x)
\]

\[
= 1 - \frac{1}{\sqrt{\pi}} \left[ \frac{e^{-x^2}}{x} - \frac{1}{2} F_1(x) \right]
\]

\[
= 1 - \frac{1}{\sqrt{\pi}} \left[ e^{-x^2} \left( \frac{1}{x} - \frac{1}{2x^3} \right) + \frac{1 \cdot 3}{2^2} F_2(x) \right]
\]

\[
= \ldots
\]

\[
= 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{N-1} (-1)^n \frac{(2n-1)!!}{2^n x^{2n+1}} + R_N(x),
\]

where \((2n-1)!! = 1 \cdot 3 \cdots (2n-1)\), and

\[
R_N(x) = (-1)^N \frac{1}{\sqrt{\pi}} \frac{(2N-1)!!}{2^N} F_N(x).
\]

The series diverges for any value of \( x \) because the ratio between the \((n+1)\)-th and the \(n\)-th term

\[
\left| \frac{u_{n+1}}{u_n} \right| = \frac{(2n+1)!!}{2^{n+1} x^{2n+3}} \times \frac{2^n x^{2n+1}}{(2n-1)!!} = \frac{2n + 1}{2n^2}, \quad (14)
\]

is larger than 1 whenever \(2n + 1 > 2x^2\). In spite of that,

\[
|R_N(x)| \leq C_N \left| \int_{x^2}^{\infty} s^{-N-1/2} e^{-s} \, ds \right|
\]

\[
\leq C_N \frac{x^{2N+1}}{x^{2N+1}} \int_{x^2}^{\infty} e^{-s} \, ds
\]

\[
\leq C_N \frac{e^{-x^2}}{x^{2N+1}},
\]

where \(C_N = (2N-1)!!/(2^N \sqrt{\pi})\). Thus,

\[
\text{erf}(x) - \left[ 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{N-1} (-1)^n \frac{(2n-1)!!}{2^n x^{2n+1}} \right] = O\left( \frac{e^{-x^2}}{x^{2N+1}} \right), \quad \text{as } x \to \infty,
\]

and the series gives an asymptotic expansion of \( \text{erf}(x) \) for large \( x \):

\[
\text{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n x^{2n+1}}, \quad \text{as } x \to \infty. \quad (15)
\]

Only the first two terms of the expansion (15) suffice to have an approximation to \( \text{erf}(3) \) with an accuracy of \(10^{-5}\). The series, nevertheless, does not provide an arbitrary accurate approximation to \( \text{erf}(x) \) and we cannot improve
the approximation at will by adding more terms. From (14) it follows that the ratio between two successive terms of the series becomes larger than 1 as $n \gtrsim x^2$. Hence, the asymptotic series (15) has an optimal truncation for $N$ equal to the largest integer smaller than $x^2$.

In this example we have encountered an asymptotic expansion of the form

$$f(x) \sim g(x) \sum_{n=0}^{\infty} a_n \varphi_n(x), \quad \text{as } x \to x_0,$$

where the function $g(x)$ is bounded as $x \to x_0$. In the example $g(x) \propto e^{-x^2}$, clearly bounded as $x \to \infty$. This reflects the non uniqueness of gauge functions: since $g(x)$ is bounded as $x \to x_0$ the functions $\varphi_n(x)$ and $\tilde{\varphi}_n(x) = g(x) \varphi_n(x)$ form two different asymptotic sequence as $x \to x_0$. This provides additional flexibility, but it should be used with care because differences might be numerically relevant for $x$ not too close to $x_0$.

**Example 2.9.** The asymptotic expansion of the function $f(x) = 1/(1 - x)$ as $x \to 0$ with gauge function $\varphi_n(x) = x^n$ is:

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n, \quad \text{as } x \to 0.$$  \hspace{1cm} (16)

The series with gauge functions $\tilde{\varphi}_n(x) = (1 + x) x^n$ is also an asymptotic expansion of $f(x)$ as $x \to 0$ because the function $g(x) = 1 + x$ is bounded as $x \to 0$:

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} (1 + x) x^n, \quad \text{as } x \to 0.$$ \hspace{1cm} (17)

Both series (16) and (17) provide an asymptotic expansion of $f(x) = 1/(1 - x)$ as $x \to 0$, but the approximation to $f(x)$ for $x$ close to 0 are different. The difference between the two approximations decreases as $x$ approaches 0, but for small finite value of $x$ it might be numerically relevant. For example, for $x = 0.1$ the first three terms of (16) give the approximantion 1.11111\ldots, while the first three terms of (17) give 1.221.

**2.4. Asymptotic power expansion**

Asymptotic expansions of the form

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{as } x \to x_0,$$

are called *asymptotic power expansions*, or *asymptotic power series*, and are among the most common and useful asymptotic expansions.
One of the fundamental notions of real analysis is the Taylor expansion of infinitely differentiable $C^\infty$ functions. If $f(x)$ is a smooth, infinitely differentiable $C^\infty$ function in a neighborhood of $x = x_0$, then the Taylor theorem ensures that

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + o(|x - x_0|^N), \quad \text{as } x \to x_0.$$  \hspace{1cm} (18)

Hence, $f(x)$ has the asymptotic power expansion

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad \text{as } x \to x_0.$$  \hspace{1cm} (18)

If, and only if, $f(x)$ is analytic at $x = x_0$ the asymptotic power series converges to $f(x)$ in a neighborhood $|x - x_0| < r$ of $x_0$ with $r > 0$. In this case the series gives the usual Taylor series representation of the function about $x_0$. Nevertheless, the Taylor expansion of $f(x)$ about the point $x = x_0$ does not provide an asymptotic expansion of $f(x)$ as $x \to x_1 \neq x_0$, even if convergent for $|x_1 - x_0| < r$; partial sums of the Taylor expansion do not usually give good approximations to $f(x)$ as $x \to x_1 \neq x_0$.

If $f(x)$ is $C^\infty$ but not analytic at $x = x_0$ the series (18) need not converge in any neighborhood of $x_0$, or if it does converge, its sum need not necessarily be $f(x)$. See Example 2.5, where the function $g(x)$ is $C^\infty$ but not analytic at $x = 0$ for any $c \neq 0$.

For asymptotic power expansions the following theorem, due to Borel, provides, in some sense, the inverse of the Taylor theorem. The Borel theorem states that for any sequence of real numbers $a_n$ there exists an infinitely differentiable $C^\infty$ function $f(x)$ such that $a_n = f^{(n)}(x_0)/n!$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{as } x \to x_0.$$  \hspace{1cm} (18)

The function $f(x)$ need not to be unique. Hence, unlike convergent Taylor series, there are no restrictions on the growth rate of the coefficients $a_n$ of an asymptotic power series, as shown in the following example.

**Example 2.10.** Consider the integral (Euler, 1754):

$$f(x) = \int_{0}^{\infty} dt \frac{e^{-t}}{1 + xt},$$  \hspace{1cm} (19)

defined for $x \geq 0$. Integrating by parts we get

$$f(x) = - \int_{0}^{\infty} dt \frac{1}{1 + xt} \frac{d}{dt} e^{-t} = 1 - x \int_{0}^{\infty} dt \frac{e^{-t}}{(1 + xt)^2}.$$  \hspace{1cm} (19)

One more integration by parts gives,

$$f(x) = 1 - x + 2x^2 \int_{0}^{\infty} dt \frac{e^{-t}}{(1 + xt)^3},$$  \hspace{1cm} (19)
and one more,

\[ f(x) = 1 - x + 2x^2 - 2 \cdot 3x^3 \int_0^\infty dt \frac{e^{-t}}{(1+xt)^4}. \]

Thus, after N integration by parts:

\[ f(x) = \sum_{n=0}^{N-1} (-1)^n n! x^n + R_N(x), \]

where

\[ R_N(x) = (-1)^N N! x^N \int_0^\infty dt \frac{e^{-t}}{(1+xt)^{N+1}}. \]

The coefficients of the power series grow as the factorial of n and the ratio between two successive terms \(|u_n/u_{n-1}| = nx\) becomes larger than 1 as \(n > 1/x\).

The series does not converge for any \(x > 0\), nevertheless, for any \(N\)

\[ |R_N(x)| \leq N! x^N \int_0^\infty dt \frac{e^{-t}}{(1+xt)^N} \leq N! x^N \int_0^\infty dt e^{-t} = N! x^N. \]  \hspace{1cm} (20)

Thus \(R_N(x) = o(x^{N-1})\) as \(x \to 0^+\), and

\[ \int_0^\infty dt \frac{e^{-t}}{1+xt} \sim \sum_{n=0}^{\infty} (-1)^n n! x^n, \text{ as } x \to 0^+. \]

The power series is not convergent because the integrand in (19) has a nonintegrable singularity at \(t = -1/x\) and the integral is not defined for \(x < 0\). A convergent power series about \(x = 0\) would converge in a finite interval centred at \(x = 0\), hence also for negative \(x\).

Since the asymptotic power series does not converge, its partial sum does not provide an arbitrary accurate approximation to the integral (19) for any fixed \(x > 0\). However, from (20) it follows that for any fixed \(N\) the error is polynomial in \(x\). What is the optimal truncation of the series, that is the one which gives the best approximation? The ratio between two successive terms \(|u_n/u_{n-1}| = nx\) grows with \(n\) and becomes larger than 1 as \(n > 1/x\). The best approximation occurs when \(N \sim [1/x]\), for which \(R_N \sim (1/x)! x^{1/x}\). Using the Stirling’s formula \(N! \sim \sqrt{2\pi N} N^N e^{-N}\) valid for \(N \gg 1\), the error of the optimal truncation reads:

\[ R_N(x) \sim \sqrt{\frac{2\pi}{x}} e^{-1/x}, \text{ as } x \to 0^+. \]

Thus, while for arbitrary \(N\) the partial sum is polynomially accurate in \(x\), the optimal truncated sum is exponentially accurate.

2.5. Stokes Phenomenon

The asymptotic expansion as \(z \to z_0\) of complex function \(f(z)\) with an essential singularity in \(z = z_0\) is typically valid only in wedge-shaped regions
\[ \alpha < \arg(z - z_0) < \beta, \] and the function has different asymptotic expansion in different wedges. For definitiveness we shall discuss the case \( z_0 = \infty \), however the same scenario appears for essential singularities at any \( z_0 \in \mathbb{C} \).

The change of the form of the asymptotic expansion across the boundaries of the wedges is called Stokes phenomenon. The origin of the Stokes phenomenon can be understood as follows. If the function \( g(z) \) is asymptotically equal to the function \( f(z) \) as \( z \to \infty \) this means that \( f(z) = g(z) + h(z) \) with \( h(z) \) subdominant compared with \( g(z) \) as \( z \to \infty \). As \( z \) approaches the boundary of the wedge of validity of the expansion the subdominant \( h(z) \) grows in magnitude compared to the dominant term \( g(z) \). At the boundary of the wedge the functions \( h(z) \) and \( g(z) \) are of the same order of magnitude and the distinction between dominant and subdominant disappears. As \( z \) crosses the boundary the functions \( h(z) \) and \( g(z) \) exchange their role and on the other side \( g(z) \ll h(z) \) as \( z \to \infty \). This exchange is the Stokes phenomenon, and it is typical of exponential functions.

**Example 2.11.** Consider the complex function

\[ f(z) = \sinh z = \frac{e^z - e^{-z}}{2} \]

In the half plane \( \Re z > 0 \) both \( \sinh z \) and \( e^z \) grows exponentially as \( z \to \infty \) and

\[ \sinh z \sim \frac{1}{2} e^z, \quad \text{as } z \to \infty, \ \Re z > 0. \]

In this region the difference \( h(z) = -\frac{1}{2} e^{-z} \) between \( \sinh z \) and \( g(z) = \frac{1}{2} e^z \) is transcendentally small as \( z \to \infty \).

In the half plane \( \Re z < 0 \) the function \( h(z) = -\frac{1}{2} e^{-z} \) is no longer transcendentally small, and

\[ \sinh z \sim -\frac{1}{2} e^{-z}, \quad \text{as } z \to \infty, \ \Re z < 0. \]

In the region \( \Re z < 0 \) the function \( g(z) = \frac{1}{2} e^z \) is subdominant.

The two leading asymptotic expansions \( \frac{1}{2} e^z \) and \( -\frac{1}{2} e^{-z} \) switch from being dominant to subdominant on the line \( \Re z = 0 \) (anti-Stokes lines), where they are purely oscillatory. They are most **unequal** in magnitude on the line \( \Im z = 0 \) (Stoke lines), where they are purely real and either exponential growing or decreasing as \( z \to \infty \).

The Stokes and anti-Stokes lines are not unique and do not really have a precise definition because the region where the function \( f(z) \) has some asymptotic expansion is rather vague reflecting the non uniqueness of gauge functions. In the above example, we could equally well take as Stokes line any line \( \Im z = a \). As a general definition, the anti-Stokes lines are roughly where some term in the asymptotic expansion changes from increasing to decreasing. Hence, anti-Stokes lines bound regions where the function has different asymptotic behaviors. The Stokes lines are lines along which some term approaches infinity or zero fastest.
3. Elementary Operations on Poicaré type Asymptotic Expansions

In singular perturbation methods we usually assume some form of the expansions, substitute them into the equations, and perform elementary operations such as addition, subtraction, exponentiation, integration and differentiation. Although the expansions are only formal and can be divergent, these operations are generally carried out without justification. Nevertheless conditions under which these operations are justified can be given.

In the following we shall discuss some elementary properties of the Poincaré type expansion (8) of functions \( f(x; \epsilon) \). If not specified the gauge functions \( \varphi_n(\epsilon) \) are generic.

3.1. Equating Coefficients.

It is not strictly correct to write
\[
\sum_{n=0}^{\infty} a_n(x) \varphi_n(\epsilon) \sim \sum_{n=0}^{\infty} b_n(x) \varphi_n(\epsilon), \quad \text{as } \epsilon \to 0 \tag{21}
\]
because asymptotic series can only be asymptotic to functions. However what we mean with (21) is that
\[
\sum_{n=0}^{\infty} a_n(x) \varphi_n(\epsilon) \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(x) \varphi_n(\epsilon),
\]
are asymptotic to functions which are asymptotically equivalent with respect to the gauge functions \( \varphi_n(\epsilon) \) as \( \epsilon \to 0 \). That is, they differ by terms \( o[\varphi_n(\epsilon)] \) as \( \epsilon \to 0 \) for all \( n \). The uniqueness of the Poincaré type expansions given the gauge functions \( \varphi_n(\epsilon) \) then implies that the coefficients of \( \varphi_n(\epsilon) \) in the two series must agree. Hence, we can equate the coefficient of the two expansions in (21) and say that from (21) it follows that \( a_n(x) = b_n(x) \).

3.2. Addition and Subtraction.

Addition and subtraction are in general justified, thus
\[
\alpha f(x; \epsilon) + \beta g(x; \epsilon) \sim \sum_{n=0}^{\infty} \left[ \alpha a_n(x) + \beta b_n(x) \right] \varphi_n(\epsilon), \quad \text{as } \epsilon \to 0.
\]
where \( \alpha \) and \( \beta \) are constants and \( a_n(x) \) and \( b_n(x) \) the expansion coefficients of \( f \) and \( g \), respectively. The proof is simple.

3.3. Multiplication and Division.

Multiplication of asymptotic expansions is justified only if the result is still an asymptotic expansion. The formal product of \( \sum a_n \varphi_n \) and \( \sum b_n \varphi_n \) generates all products \( \varphi_n \varphi_m \) of gauge functions and generally it is not possible to rearrange them into an asymptotic sequence. Multiplication is justified only for asymptotic sequences \( \varphi_n \) such that the products \( \varphi_n \varphi_m \) lead to an asymptotic
sequence, or posses asymptotic expansions. An important class of asymptotic
expansions with this property are asymptotic power expansions. In this case
the multiplication is straightforward. Suppose $f(x; \epsilon) \sim \sum_{n=0}^{\infty} a_n(x) \epsilon^n$ and
$g(x; \epsilon) \sim \sum_{n=0}^{\infty} b_n(x) \epsilon^n$ as $\epsilon \to 0$, then

$$f(x; \epsilon) g(x; \epsilon) \sim \sum_{n=0}^{\infty} c_n(x) \epsilon^n \quad \text{as} \quad \epsilon \to 0,$$

where

$$c_n(x) = \sum_{m=0}^{n} a_m(x) b_{n-m}(x). \quad (22)$$

Proof. Using (22), and rearranging the sums, the reminder $f g - \sum_{n=0}^{N} c_n \epsilon^n$
takes the form,

$$fg - \sum_{n=0}^{N} \sum_{m=0}^{n} a_m b_{n-m} \epsilon^n = fg - \sum_{m=0}^{N} a_m \epsilon^m \sum_{n=m}^{N} b_{n-m} \epsilon^{n-m}$$

$$= g \left[ f - \sum_{m=0}^{N} a_m \epsilon^m \right]$$

$$+ \sum_{m=0}^{N} a_m \epsilon^m \left[ g - \sum_{p=0}^{N-m} b_p \epsilon^p \right].$$

Since as $\epsilon \to 0$

$$f - \sum_{n=0}^{N} a_n \epsilon^n = o(\epsilon^N),$$

and moreover $g(x; \epsilon) - b_0(x) = O(\epsilon)$, both terms in the last equality are $o(\epsilon^N)$
as $\epsilon \to 0$. Thus,

$$f(x; \epsilon) g(x; \epsilon) - \sum_{n=0}^{N} c_n(x) \epsilon^n = o(\epsilon^N), \quad \text{as} \quad \epsilon \to 0,$$

concluding the proof.

The division of asymptotic expansions can be handled treating it as the
inverse of the multiplication, hence division is not justified in general. In the
particular case of asymptotic power expansions assuming $b_0(x) \neq 0$, and using
(22), it follows that:

$$\frac{f(x; \epsilon)}{g(x; \epsilon)} \sim \sum_{n=0}^{\infty} d_n(x) \epsilon^n \quad \text{as} \quad \epsilon \to 0,$$
with \( d_0(x) = a_0(x)/b_0(x) \) and
\[
d_n(x) = \frac{1}{b_0(x)} \left[ a_n(x) - \sum_{m=0}^{n-1} d_m(x) b_{n-m}(x) \right], \quad n \geq 1.
\]

If the expansion coefficients \( a_n(x) \) and/or \( b_n(x) \) are functions of \( x \) the division of uniformly valid asymptotic expansions might lead to a non-uniformly valid expansion. When this occurs the division is not justified anymore. The Example 2.7, with \( f(x; \epsilon) \sim 1 \) and \( g(x; \epsilon) \sim x + \epsilon \) as \( \epsilon \to 0^+ \), illustrates this point.

3.4. Exponentiation.

Exponentiation of asymptotic expansions is generally not justified. As for multiplication, exponentiation produces all products of gauge functions. Thus the exponentiation can be justified only for asymptotic expansions such that the products \( \varphi_n \varphi_m \) form an asymptotic sequence, or posses asymptotic expansions. Moreover, if the coefficients of the asymptotic expansion are functions of \( x \) the exponentiation might results in a non-uniform expansion. For instance, Example 2.7 shows that the asymptotic expansion of \((x+\epsilon)^{-1}\) as \( \epsilon \to 0^+ \) is not uniformly valid for \( x \geq 0 \).

**Example 3.1.** Consider the function \( f(x; \epsilon) \) with the uniform asymptotic expansion
\[
f(x; \epsilon) \sim x + \epsilon \quad \text{as} \quad \epsilon \to 0^+,
\]
in the domain \( D : x \geq 0 \). Then
\[
\sqrt{f(x; \epsilon)} \sim \sqrt{x + \epsilon} \sim \sqrt{x} \left[ 1 + \frac{\epsilon}{2x} - \frac{\epsilon^2}{8x^2} + \ldots \right] \quad \text{as} \quad \epsilon \to 0^+.
\]
This expansion is valid uniformly in the domain \( D_0 : 0 < x_0 \leq x \), where \( x_0 \) is a positive constant, but not in the domain \( D \). The uniform validity of the asymptotic expansion breaks down when \( x/\epsilon = O(1) \), and indeed when \( x = 0 \) the expansion has a completely different form:
\[
\sqrt{f(0; \epsilon)} \sim \sqrt{\epsilon} \quad \text{as} \quad \epsilon \to 0^+.
\]

3.4.1. Integration.

Asymptotic expansions can be generally integrated term-by-term with respect to the expansion parameter \( \epsilon \) and/or the variable \( x \).

If \( f(x; \epsilon) \) and the gauge functions \( \varphi_n(\epsilon) \) are integrable functions of \( \epsilon \) in the interval \( I : \epsilon \in [0; \epsilon_0] \) where \( \epsilon_0 \) is constant, then term-by-term integration is justified and
\[
\int_0^\epsilon d\epsilon' f(x; \epsilon') \sim \sum_{n=0}^\infty a_n(x) \int_0^\epsilon d\epsilon' \varphi_n(\epsilon') \quad \text{as} \quad \epsilon \to 0.
\]
Proof. The proof is straightforward. Since $f - \sum_{n=0}^{N} a_n \varphi_n = o(\varphi_N)$ as $\epsilon \to 0$, then:

$$
\int_{0}^{\epsilon} d\epsilon' f(x; \epsilon') - \sum_{n=0}^{N} a_n(x) \int_{0}^{\epsilon} d\epsilon' \varphi_n(\epsilon') = \int_{0}^{\epsilon} d\epsilon' \left[ f(x; \epsilon') - \sum_{n=0}^{N} a_n(x) \varphi_n(\epsilon') \right] \\
= \int_{0}^{\epsilon} d\epsilon' o[\varphi_N(\epsilon')] \\
= o\left[ \int_{0}^{\epsilon} d\epsilon' \varphi_N(\epsilon') \right], \quad \text{as } \epsilon \to 0^+.
$$

In the case of asymptotic power expansions

$$
f(x; \epsilon) \sim \sum_{n=0}^{\infty} a_n(x) \epsilon^n \quad \text{as } \epsilon \to 0,
$$

termwise integration is always justified and leads to

$$
\int_{0}^{\epsilon} d\epsilon' f(x; \epsilon') \sim \sum_{n=0}^{\infty} \frac{a_n(x)}{n+1} \epsilon^{n+1}, \quad \text{as } \epsilon \to 0.
$$

On the contrary, the first two terms of the asymptotic expansion

$$
f(x; \epsilon) \sim \sum_{n=0}^{\infty} a_n(x) \epsilon^{-n}, \quad \text{as } \epsilon \to +\infty,
$$

are not integrable at $\epsilon = +\infty$, and termwise integration of the series is not justified. Nevertheless,

$$
\int_{\epsilon}^{\infty} dt \left[ f(x; t) - a_0(x) - a_1(x) t^{-1} \right] \sim \sum_{n=2}^{\infty} \frac{a_n(x)}{n-1} \epsilon^{1-n}, \quad \text{as } \epsilon \to +\infty.
$$

The proof is straightforward and is left as exercise.

The extension to asymptotic expansions with gauge functions $\varphi_n(\epsilon) = \epsilon^{\alpha_n}$ as $\epsilon \to 0^+$, or $\varphi_n(\epsilon) = \epsilon^{-\alpha_n}$ as $\epsilon \to +\infty$, with $\alpha_{n+1} > \alpha_n$ is trivial.

Similarly, if $f(x; \epsilon)$ and $a_n(x)$ are integrable functions of $x$ in some interval $I : x \in [x_0, x_1]$, then for any $x \in I$

$$
\int_{x_0}^{x} dx' a_n(x') \sim \sum_{n=0}^{\infty} \varphi_n(\epsilon) \int_{x_0}^{x} dx' a_n(x'), \quad \text{as } \epsilon \to 0.
$$

The proof is straightforward.
3.5. Differentiation.

Termwise differentiation of uniform asymptotic expansions, either with respect to the perturbation parameter $\epsilon$ or with respect to the variable $x$, might lead to non-uniformly valid asymptotic expansions. When this occurs the differentiation of the asymptotic expansion term-by-term is not justified.

**Example 3.2.** Consider the function $f(x; \epsilon)$ with the uniform asymptotic expansion

$$f(x; \epsilon) \sim 1 + \epsilon \sqrt{x} \quad \text{as} \quad \epsilon \to 0^+,$$

in the domain $D : x \geq 0$. Then

$$\frac{\partial}{\partial x} f(x; \epsilon) \sim \frac{\epsilon}{2\sqrt{x}} \quad \text{as} \quad \epsilon \to 0^+. $$

This expansion is valid uniformly in the domain $D_0 : 0 < x_0 \leq x$, where $x_0$ is a positive constant, but not in the domain $D$.

In addition to this, subdominant terms might become relevant under differentiation as illustrated by the following example.

**Example 3.3.** The functions $f(x)$ and

$$g(x) = f(x) + e^{-x^{-2}} \sin e^{-x^{-2}},$$

have the same asymptotic power expansion as $x \to 0$ because $e^{-x^{-2}}$ is transcendentally small with respect to any power $x^n$ as $x \to 0$ and the last term is subdominant. However their derivatives $f'(x)$ and

$$g'(x) = f'(x) + 2 \frac{e^{-x^{-2}}}{x^3} \sin e^{-x^{-2}} - \frac{2}{x^3} \cos e^{-x^{-2}},$$

do not have the same power expansion as $x \to 0$ because $x^{-3}$ is not subdominant to $x^n$ as $x \to 0$.

To differentiate term-by-term the asymptotic expansion

$$f(x; \epsilon) \sim \sum_{n=0}^{\infty} a_n(x) \varphi_n(\epsilon), \quad \text{as} \quad \epsilon \to 0,$$  \hspace{1cm} (23)

some additional conditions are needed. For instance, assume it is known that the partial derivative $\partial f(x; \epsilon)/\partial \epsilon$ exists and possess an asymptotic expansion as $\epsilon \to 0$ with some gauge functions $\psi_n(\epsilon)$ and expansion coefficients $a_n(x)$. If $\partial f(x; \epsilon)/\partial \epsilon$ and $\psi_n(\epsilon)$ are integrable functions of $\epsilon$ in the interval $I : 0 \leq \epsilon \leq \epsilon_0$, then the asymptotic expansion can be integrated term-by-term with respect to $\epsilon$ leading to the asymptotic expansion (23) of $f(x; \epsilon)$ with $\varphi_n(\epsilon) = \int_0^{\epsilon} d\epsilon \psi_n(\epsilon)$. Under these conditions termwise differentiation with respect to $\epsilon$ of the asymptotic expansion (23) is justified and leads to:

$$\frac{\partial f(x; \epsilon)}{\partial \epsilon} \sim \sum_{n=0}^{\infty} a_n(x) \varphi'_n(\epsilon), \quad \text{as} \quad \epsilon \to 0.$$
Similarly, if it is known that \( \partial f(x; \epsilon)/\partial x \) exists and possess an asymptotic expansion as \( \epsilon \to 0 \) with gauge functions \( \phi_n(\epsilon) \) and expansion coefficients \( b_n(x) \). And, moreover, \( \partial f(x; \epsilon)/\partial x \) and \( b_n(x) \) are integrable functions of \( x \) in the interval \( I : x_0 \leq x \leq x_1 \), then the asymptotic expansion can be integrated term-by-term with respect to \( x \) obtaining the asymptotic expansion (23) of \( f(x; \epsilon) \) with \( a_n(x) = \int_0^x dx' b_n(x') \) for \( x \in I \). Thus, if these conditions are satisfied termwise differentiation of the asymptotic expansion (23) with respect to \( x \) is justified and leads to:

\[
\frac{\partial f(x; \epsilon)}{\partial x} \sim \sum_{n=0}^{\infty} a_n'(x) \phi_n(\epsilon), \quad \text{as } \epsilon \to 0.
\]

We shall came back on this point when discussing the asymptotic expansion of the solutions of differential equations.

4. Asymptotic Expansion of Integrals

Asymptotic expansion of integrals is a collection of techniques originally developed because many special functions used in Physics and Applied Mathematics have integral representations. Nowadays asymptotic expansion of integrals find applications in different fields, such as Statistical Mechanics and Field Theory. We shall not discuss asymptotic expansion of integrals here, rather we discuss two examples to illustrate the use of asymptotic expansions.


Consider the function \( f(x) \) defined by the Stieltjes integral:

\[
f(x) = \int_0^\infty dt \frac{\rho(t)}{1 + xt}, \quad x \geq 0
\]

where \( \rho(t) \) is a generic non-negative function:

\[\rho(t) \geq 0, \quad t \geq 0.\]

We have already encountered this type of integral in Example 2.10 with \( \rho(t) = e^{-t} \). There we obtained the asymptotic power expansion of \( f(x) \) as \( x \to 0^+ \) by successive integration by parts. For a generic function \( \rho(t) \) this procedure cannot be applied. Nevertheless, an asymptotic power expansion of \( f(x) \) as \( x \to 0^+ \) can be obtained by substituting into the integrand the asymptotic powers expansion

\[
\frac{1}{1 + xt} \sim \sum_{n=0}^{\infty} (-xt)^n, \quad \text{as } x \to 0^+,
\]

and integrating term-by-term. Thus, if the moments

\[
a_n = \int_0^\infty dt \rho(t) t^n \quad n \geq 0
\]

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exists and are finite for any \( n \), the function \( f(x) \) as \( x \to 0^+ \) has the asymptotic expansion:

\[
f(x) \sim \sum_{n=0}^{\infty} (-1)^n a_n x^n, \quad \text{as } x \to 0^+, \tag{26}
\]
called Stieltjes Series. The series does not converge for any function \( \rho(t) \). A convergent power series in \( x \) converges inside a disk of finite radius centered at \( x = 0 \). The Stieltjes integral (24) is not defined for negative \( x \) because the singularity at \( t = -1/x \) is not integrable; thus, the Stieltjes Series (26) cannot be convergent.

**Proof.** The proof that series (26) is an asymptotic power expansion as \( x \to 0^+ \) is simple. By using (25) and the identity

\[
\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z},
\]
valid for any \( N \), we have

\[
f(x) - \sum_{n=0}^{N} (-1)^n a_n x^n = \int_{0}^{\infty} dt \rho(t) \left[ \frac{1}{1 + xt} - \sum_{n=0}^{N} (\neg x t^n) \right]
\]

\[
= \int_{0}^{\infty} dt \rho(t) \left[ \frac{1}{1 + xt} - \frac{1 - (\neg x t)^{N+1}}{1 + xt} \right]
\]

\[
= \int_{0}^{\infty} dt \frac{\rho(t)}{1 + xt} (-xt)^{N+1}.
\]

Then

\[
\left| f(x) - \sum_{n=0}^{N} (-1)^n a_n x^n \right| \leq x^{N+1} \int_{0}^{\infty} dt \rho(t) t^{N+1} = a_{N+1} x^{N+1}
\]

and

\[
f(x) - \sum_{n=0}^{N} (-1)^n a_n x^n = o(x^N), \quad \text{as } x \to 0^+,
\]

which concludes the proof. \( \square \)

### 4.2. Gamma Function

The gamma function \( \Gamma(\nu) \) is defined for all complex numbers \( \nu \) with real positive part as

\[
\Gamma(\nu) = \int_{0}^{\infty} dt t^{\nu-1} e^{-t}, \quad \Re \nu > 0. \tag{27}
\]

The gamma function is then extended via an analytic continuation to the whole complex plane except to non-positive integers, where it has simple poles. For simplicity we consider the case of real positive large \( \nu \).
By using the functional relation \( \nu \Gamma(\nu) = \Gamma(1 + \nu) \) and introducing the new variable \( t = \nu(y + 1) \), \( \Gamma(\nu) \) can be written as

\[
\Gamma(\nu) = \frac{1}{\nu} \Gamma(1 + \nu) = \nu^{-\nu} e^{-\nu} \int_{-1}^{\infty} dy e^{\nu h(y)},
\]

where

\[
h(y) = \ln(1 + y) - y.
\]

The function \( h(y) \) has a global maximum at \( y = 0 \), see Figure 1. When \( \nu \gg 1 \)

only the neighborhood of \( y = 0 \) contributes significantly to the integral because contributions from \( y \) not in the neighborhood 0 are exponentially depressed. Then \( \Gamma(\nu) \) can be estimated for \( \nu \gg 1 \) by restricting the integration to the neighborhood of \( y = 0 \), where \( h(y) \) attains its maximum value (Laplace’s method). We thus divide the interval of integration into three regions:

\[
\int_{-1}^{\infty} dy e^{\nu h(y)} = \int_{-\epsilon}^{\epsilon} dy e^{\nu h(y)} + \int_{-\epsilon}^{-\epsilon} dy e^{\nu h(y)} + \int_{\epsilon}^{\infty} dy e^{\nu h(y)},
\]

were \( \epsilon \ll 1 \) is arbitrary, but sufficiently small so that in the interval \( -\epsilon \leq y \leq \epsilon \) the function \( h(y) \) can be replaced by the Maclaurin expansion:

\[
h(y) = -\frac{1}{2} y^2 + \frac{1}{3} y^3 - \frac{1}{4} y^4 + O(y^5).
\]

The parameter \( \epsilon \) may depend on \( \nu \), but for the moment it is sufficient to know that \( \epsilon \ll 1 \) so that (28) can be used. The functional dependence of \( \epsilon \) on \( \nu \) is
fixed later to make the whole calculation consistent. As \( \nu \to +\infty \) the first and third integrals are of the order

\[
\int_{-\epsilon}^{\epsilon} dy e^{\nu h(y)} = O\left(e^{\nu(-\epsilon)}\right) = O\left(e^{-\nu \epsilon^2/2}\right),
\]

\[
\int_{-\epsilon}^{\epsilon} dy e^{\nu h(y)} = O\left(e^{\nu(\epsilon)}\right) = O\left(e^{-\nu \epsilon^2/2}\right).
\]

Provided \( \epsilon \sqrt{\nu} \gg 1 \), i.e., \( \nu^{-1/2} < \epsilon \ll 1 \), their contribution is exponentially small as \( \nu \gg 1 \).

Replacing \( h(y) \) into the second integral by the expansion (28), and introducing the variable \( x = y \sqrt{\nu} \), then:

\[
\int_{-\epsilon}^{\epsilon} dy e^{\nu h(y)} \sim \nu^{-1/2} \int_{-\epsilon}^{\epsilon} \sqrt{\nu} dx e^{-\frac{1}{4} x^2 + \frac{1}{2} \epsilon x^3 - \frac{1}{12} x^4 + O(x^5/\nu^{3/2})} + O(e^{-\nu \epsilon^2/2}).
\]  

(29)

Now we use the arbitrariness of \( \epsilon \) and take \( \epsilon = \nu^{-a} \) with \( 1/3 < a < 1/2 \). This choice not only ensures that \( \epsilon \ll 1 \) and \( \epsilon \sqrt{\nu} \gg 1 \) as \( \nu \to \infty \), but also that for \( |x| \leq \epsilon \sqrt{\nu} \):

\[
\frac{2^m}{\nu^{m/2 - 1}} \ll \frac{2^n}{\nu^{n/2 - 1}} \ll 1 \quad m > n, \quad n, m = 3, 4, 5, \ldots.
\]

The exponential under the integral in (29) is then of the form \( \exp[-x^2/2 + g_v(x)] \) with \( g_v(x) \ll 1 \) as \( \nu \to +\infty \) in the whole domain of integration \([-\epsilon \sqrt{\nu}, \epsilon \sqrt{\nu}]\).

Expanding the exponential \( \exp[g_v(x)] \) in powers of \( g_v(x) \) we obtain:

\[
\int_{-\epsilon}^{\epsilon} dy e^{\nu h(y)} \sim \nu^{-1/2} \int_{-\epsilon \sqrt{\nu}}^{\epsilon \sqrt{\nu}} dx e^{-\frac{1}{4} x^2} \left[1 + g_v(x) + \frac{1}{2} g_v(x)^2 + \ldots\right] + O(e^{-\nu \epsilon^2/2}).
\]

The integration limits \( \pm \epsilon \sqrt{\nu} \) can be safely extended to \( \pm \infty \) with an error of order \( O(e^{-\nu \epsilon^2/2}) \), exponentially small as \( \nu \to +\infty \). Thus, introducing the Gaussian average:

\[
\langle f \rangle = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{4} x^2} f(x),
\]

the asymptotic expansion of the function \( \Gamma(\nu) \) as \( \nu \to +\infty \) can be written as

\[
\Gamma(\nu) \sim \nu^{-1/2} e^{-\nu \sqrt{2\pi}} \left[1 + \langle g_v \rangle + \frac{1}{2} \langle g_v^2 \rangle + \frac{1}{3!} \langle g_v^3 \rangle + \ldots\right],
\]

where

\[
g_v(x) = \frac{1}{3\sqrt{\nu}} x^3 - \frac{1}{4\nu} x^4 + O(x^5/\nu^{3/2}).
\]

The averages can be evaluated using the well known result for the moments of the Gaussian distribution of zero average and variance 1: \( \langle x^{2n} \rangle = (2n - 1)!! \) and \( \langle x^{2n+1} \rangle = 0 \). To the leading order we have

\[
\langle g_v \rangle = -\frac{1}{4\nu} \langle x^4 \rangle + O(1/\nu^2) = -\frac{3}{4\nu} + O(1/\nu^2),
\]

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\[ \langle g^2 \rangle = \frac{1}{9} \langle x^6 \rangle + O(1/\nu^2) = \frac{5 \cdot 3}{9} \nu + O(1/\nu^2). \]

and \( \langle g^3 \rangle = O(1/\nu^2) \), so that
\[ \Gamma(\nu) \sim \nu^{-1/2} e^{-\nu} \sqrt{2\pi} \left[ 1 + \frac{1}{12\nu} + O(1/\nu^2) \right], \quad \text{as } \nu \to +\infty, \tag{30} \]

known as Stirling’s formula, after James Stirling, even if it was first stated by Abraham de Moivre.

The proof that (30) is an asymptotic expansion of \( \Gamma(\nu) \) as \( \nu \to +\infty \) is straightforward. It is enough to compute the next terms in the expansion and estimate the error when the series is truncated to the first \( N \) terms. The contributions of order \( O(e^{-\nu \epsilon^2/2}) \) are transcendentally small with respect to any power \( \nu^{-n} \) as \( \nu \to +\infty \) and can be ignored.

The asymptotic expansion (30) of \( \Gamma(\nu) \) has been derived using the integral representation (27) and the Laplace’s method; however, the definition (27) is not the only possible definition of \( \Gamma(\nu) \). For example, an asymptotic expansion of \( \Gamma(\nu) \) as \( \nu \to +\infty \) can be obtained also from the integral representation

\[ \ln \Gamma(\nu) = \left( \nu - \frac{1}{2} \right) \ln \nu - \nu + \ln \sqrt{2\pi} + 2 \int_0^\infty dt \frac{\arctg\left( \frac{t}{\nu} \right)}{e^{2\pi t} - 1}. \]

Performing the change of variable \( t = \nu y \), one sees that as \( \nu \to +\infty \) only values of \( y \) up to \( y = O(1/\nu) \) contributes significantly to the integral; the others give just a correction \( O(e^{-2\pi \nu}) \) to the value of the integral. Then, since \( 1/\nu \ll 1 \), we can replace \( \arctg(x) \) by the asymptotic expansion (Taylor series)

\[ \arctg(x) \sim \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n - 1}, \quad x \to 0, \]

and integrate termwise the resulting asymptotic expansion:

\[ \int_0^\infty dt \frac{\arctg\left( \frac{t}{\nu} \right)}{e^{2\pi t} - 1} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n - 1) \nu^{2n-1}} \int_0^\infty dt \frac{t^{2n-1}}{e^{2\pi t} - 1}. \tag{31} \]

We have neglected the \( O(e^{-2\pi \nu}) \) terms because transcendentally small with respect to any power \( \nu^{-n} \) as \( \nu \to +\infty \). Termwise integration is justified because the coefficient of the expansion are integrable functions of \( t \). The integral in (31) is equal to

\[ \int_0^\infty dt \frac{t^{2n-1}}{e^{2\pi t} - 1} = \frac{(-1)^{n-1}}{4n} B_{2n}, \tag{32} \]

where \( B_n \) are the Bernoulli numbers given by the coefficients of the expansion

\[ \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad 0 < |z| < 2\pi. \]
Substituting (32) into (31), we have

$$\ln \Gamma(\nu) \sim \left(\nu - \frac{1}{2}\right) \ln \nu - \nu + \ln \sqrt{2\pi} + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)\nu^{2n-1}}, \quad \text{as } \nu \to +\infty. \quad (33)$$

The first terms of the series using $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, are

$$\ln \Gamma(\nu) \sim \left(\nu - \frac{1}{2}\right) \ln \nu - \nu + \ln \sqrt{2\pi} + \frac{1}{12\nu} - \frac{1}{360\nu^3} + \frac{1}{1260\nu^5} + O(\nu^{-7}), \quad \text{as } \nu \to +\infty,$$

The series (33) is not convergent, but when truncated gives and approximation of $\ln \Gamma(\nu)$ as $\nu \to +\infty$ with error of the order of the first omitted term.

The asymptotic expansion (33) differs from the expansion (30) derived using the integral representation (27); nevertheless, they are asymptotically equivalent as $\nu \to +\infty$. From (33) we have

$$\Gamma(\nu) \sim \nu^{\nu - 1/2} e^{-\nu} \sqrt{2\pi} \exp \left[ \frac{1}{12\nu} + O(\nu^{-3}) \right], \quad (34)$$

which differs from (30) by terms $O(1/\nu^2)$ as $\nu \to \infty$ because

$$\exp \left[ \frac{1}{12\nu} + O(\nu^{-3}) \right] - 1 - 1/12\nu = O(1/\nu^2) \quad \text{as } \nu \to +\infty.$$ 

For finite values of $\nu$, however, they give different approximations. For example the approximation (30) gives $\Gamma(5) = 23.99\ldots$, $\Gamma(6) = 119.99\ldots$ and $\Gamma(7) = 719.95\ldots$, while approximation (34) gives $\Gamma(5) = 24.00\ldots$, $\Gamma(6) = 120.00\ldots$ and $\Gamma(7) = 720.00\ldots$. 

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