To understand what is a layer-type problem, and why it can lead to a singular perturbation problem, consider the laminar flow of a fluid in a channel. If the fluid is inviscid, the velocity does not depend on the distance from the channel edges and the velocity profile through the channel is flat. On the contrary, if the fluid has some viscosity, it sticks on the channel wall wetting it. The velocity of the fluid is zero at the channel edges. As we move away from the wall, the velocity increases and sufficiently far from it, we recover the inviscid flat profile. The width of the region where the velocity drops to zero depends on the viscosity and mainstream velocity of the fluid at the center of the channel. For fixed mainstream velocity, the width decreases as the viscosity is decreased, vanishing in the limit of an inviscid fluid. We have then a situation where the velocity is almost constant in a large region across the channel, and rapidly changes in a small region close to the channel edges. This small region is called boundary layer. The velocity profile of the fluid is regular for any value of the viscosity. Nevertheless, if one constructs a perturbative approximation to the velocity profile for small finite values of the viscosity starting from the velocity profile of the inviscid case, then it is obvious that difficulties emerge. As odd as it may seem, this is exactly what one usually does in these cases.

Layer-type problems, where the solution of the problem is rapidly changing in a narrow region (layer), may arise in different contexts, not only in fluid dynamics. We shall consider the case of layers-type problems occurring in the solution of differential equations, and in particular of ordinary differential equations. We have already encountered one, the kicked damped linear-mass spring system. A typical feature of this type of problems is the small parameter multiplying the highest derivative, leading to the reduction of the number of initial or boundary conditions that can be simultaneously satisfied by the unperturbed solution. Notice, however, that this is not a necessary nor a sufficient condition to have a singular layer-type perturbation problem.

Layer-type problems can be studied using the method of the Matched Asymptotic Expansions, in which one constructs different asymptotic Poincaré type expansions, one inside and one outside the region of rapid change, and then match them together to get an uniform asymptotic expansion in the whole region of interest.
1. Matched Asymptotic Expansions

1.1. Domain of validity, overlap and matching

In order to understand why a single Poincaré type expansion typically cannot provide an uniform approximation to the solution of a layer-type problem, let us consider the function

\[ y(x; \epsilon) = e^{-x/\epsilon} + x + \epsilon, \quad 0 \leq x \leq 1, \]  

and construct an approximation to it valid for \( \epsilon \ll 1 \). The function \( y(x; \epsilon) \) has a boundary layer of width \( \delta = O(\epsilon) \) as \( \epsilon \to 0^+ \) at \( x = 0 \), where it drops rapidly from \( 1 + \epsilon \).

In a Poincaré type expansion the first term of the approximation for \( \epsilon \ll 1 \) is obtained by taking the limit \( \epsilon \to 0^+ \) with \( x > 0 \) fixed. This limit, called outer limit, leads to

\[ y_{\text{out}}(x) = x. \]  

(2)

The function \( y_{\text{out}}(x) \) is an uniform \( O(1) \) approximation to \( f(x; \epsilon) \) as \( \epsilon \to 0^+ \) on any interval \( I_{\text{out}} : 0 < x_0 \leq x \leq 1 \), where \( x_0 \) is any positive constant independent of \( \epsilon \). Indeed for any \( x > 0 \) and gauge function \( \eta(\epsilon) = 1 \),

\[ \lim_{\epsilon \to 0^+} \frac{y(x; \epsilon) - y_{\text{out}}(x)}{\eta(\epsilon)} = \lim_{\epsilon \to 0^+} \left[ e^{-x/\epsilon} + \epsilon \right] = 0, \]  

(3)

uniformly. However, \( y_{\text{out}}(x) \) is not an uniform approximation to \( f(x; \epsilon) \) on any interval that includes \( x = 0 \) because \( y(0; \epsilon) = 1 + \epsilon \) while \( y_{\text{out}}(0) = 0 \) and the condition (3) is not satisfied at \( x = 0 \).

To find an uniform approximation to \( y(x; \epsilon) \) close to \( x = 0 \) we rewrite \( y(x; \epsilon) \) in terms of the scaled, or stretched or inner, variable \( \xi = x/\epsilon \),

\[ y(\xi; \epsilon) = e^{-\xi} + \epsilon \xi + \epsilon, \quad 0 \leq \xi \leq 1/\epsilon, \]

and construct a Poincaré type expansion of \( f(\xi; \epsilon) \) for \( \epsilon \ll 1 \). Taking the limit \( \epsilon \to 0^+ \) with \( \xi \) fixed, called inner limit, leads to the first term of the expansion:

\[ y_{\text{in}}(\xi) = e^{-\xi}. \]  

(4)

The function \( y_{\text{in}}(\xi) \) is an \( O(1) \) approximation to \( y(\xi; \epsilon) \) as \( \epsilon \to 0^+ \) uniform on any interval \( I_{\text{in}} : 0 \leq \xi \leq \xi_0 < 1/\epsilon \), where \( \xi_0 \) is a constant:

\[ \lim_{\epsilon \to 0^+} \frac{y(\xi; \epsilon) - y_{\text{in}}(\xi)}{\eta(\epsilon)} = \lim_{\epsilon \to 0^+} \left[ \epsilon \xi + \epsilon \right] = 0, \]  

(5)

uniformly. The upper bound \( \xi_0 \) of the interval must be strictly smaller that \( 1/\epsilon \) because \( y(\xi = 1/\epsilon; \epsilon) = e^{-1/\epsilon} + 1 + \epsilon \) while \( y_{\text{in}}(\xi = 1/\epsilon) = e^{-1/\epsilon} \) and the condition (5) fails at \( \xi = 1/\epsilon \).

Thus neither the expansion (2) nor the expansion (4) provides an uniform \( O(1) \) approximation to \( y(x; \epsilon) \) for \( \epsilon \to 0^+ \) in the whole interval \( 0 \leq x \leq 1 \). Nevertheless, each one separately gives an uniform approximation to \( y(x; \epsilon) \), one...
“far” and one “close” to $x = 0$. Can we combine them together to construct a single approximation to $y(x; \epsilon)$ which is uniform in the whole interval $0 \leq x \leq 1$? At the first sight it looks an hopeless task because the domains of validity $I_{\text{out}}$ and $I_{\text{in}}$ of the two expansions do not overlap. Indeed $x_0$ is a finite constant independent of $\epsilon$ while, once expressed in the variable $x$, $I_{\text{in}} : 0 \leq x \leq \epsilon x_0$ shrinks to zero as $\epsilon \to 0$.

A closer inspection, however, reveals that we can extend the domains on which the two expansions are uniform so that they do overlap. This will be at the cost of having and increase of the error but, provided it remains $o(1)$ as $\epsilon \to 0^+$, they will remain uniform $O(1)$ approximations to $y(x; \epsilon)$.

The maximum value of $y(x; \epsilon) - y_{\text{out}}(x)$ on the interval $\epsilon \leq x \leq 1$ is $\epsilon^{-1} + \epsilon$. This is not $o(1)$ as $\epsilon \to 0^+$, and hence $y_{\text{out}}(x)$ is not an uniform $O(1)$ expansion on $\epsilon \leq x \leq 1$. However, the maximum value of $y(x; \epsilon) - y_{\text{out}}(x)$ on the interval $\sqrt{\epsilon} \leq x \leq 1$ is $\epsilon^{-1/\sqrt{\epsilon}} + \epsilon$, which is now $o(1)$ as $\epsilon \to 0^+$; the expansion $y_{\text{out}}(x)$ is still uniform on this interval. Thus, the domain of validity of $y_{\text{out}}(x)$ can be extended to any interval $x_0(\epsilon) \leq x \leq 1$ where $x_0(\epsilon) \to 0$ as $\epsilon \to 0^+$ but $x_0(\epsilon) \gg \epsilon$. The domain $I_1 : \epsilon \ll x_0(\epsilon) \leq x \leq 1$ is the extended domain of validity of $y_{\text{out}}(x)$.

Similarly, the maximum value of $y(\epsilon x; \epsilon) - y_{\text{in}}(x)$ on the interval $0 \leq x \leq \xi_0$ is $\epsilon x_0 + \epsilon$, so we can extend the domain of validity of $y_{\text{in}}(x)$ to any interval $0 \leq x \leq \xi_0(\epsilon)$ with $\xi_0(\epsilon) \to +\infty$ as $\epsilon \to 0^+$ provided $\xi_0(\epsilon) \ll 1/\epsilon$. Thus, $y_{\text{in}}(x)$ is an uniform $O(1)$ approximation to $y(x; \epsilon)$ for $\epsilon \to 0^+$ on the extended domain $I_0 : 0 \leq x \leq \xi_0(\epsilon) \ll 1/\epsilon$.

If $x_0(\epsilon) \ll \epsilon x_0(\epsilon)$ as $\epsilon \to 0^+$, for example $x_0(\epsilon) = \epsilon^{1/2}$ and $\xi_0(\epsilon) = \epsilon^{-2/3}$, the extended domain of validity of $y_{\text{out}}(x)$ and $y_{\text{in}}(x)$ overlap on the domain $D = I_0 \cap I_1 : x_0(\epsilon) \leq x \leq \epsilon x_0(\epsilon)$. Note that as $\epsilon \to 0^+$ the width of the domain $D$ shrinks to zero in the variable $x$ and diverges in the variable $\xi$. Intervals with moving endpoints are essential for various singular perturbation techniques.

The point is that if two asymptotic expansions have overlapping domains of validity, they can be matched. The technique of matched uniform asymptotic expansions consists in constructing uniform asymptotic expansions with overlapping domains of validity and matching them to obtain an uniform asymptotic expansion on the whole domain of interest.

Asymptotic expansions are matched using the limiting process called intermediate matching. Limiting processes are an important tool of matched asymptotic expansions. In our example we have already seen the outer and inner limit processes used to construct the uniform asymptotic expansions of $y(x; \epsilon)$ sufficiently far and sufficiently close to $x = 0$. To perform the intermediate matching we have to generalize the limiting process defining the $\delta$-limit process

$$
\lim_{\epsilon \to 0^+} y(x; \epsilon) \overset{\text{def}}{=} \lim_{\substack{\epsilon \to 0^+ \\
\zeta_\delta \text{ fixed} \neq 0}} y(\delta \zeta_\delta; \epsilon),
$$

where $\zeta_\delta = x/\delta(\epsilon)$ and $\delta(\epsilon)$ is some function of $\epsilon$ that remains bounded as $\epsilon \to 0$.

\[1\] If $x_0(\epsilon) = \epsilon^{1/2}$ and $\xi_0(\epsilon) = \epsilon^{-2/3}$ then $D : \epsilon^{1/2} \leq x \leq \epsilon^{1/3}$ or $D : \epsilon^{-1/2} \leq x \leq \epsilon^{-2/3}$.
With this definition the inner limit used to get (4) is \( \lim_\epsilon \) with \( \zeta_\delta = \xi \). The limit \( \lim_{1} \) is the outer limit for fixed \( x \); usually the subscript “1” is omitted.

Intermediate matching is based on the following result. If \( f(x; \epsilon) \) and \( g(x; \epsilon) \) are two uniform approximations to order \( \eta(\epsilon) \) of the function \( y(x; \epsilon) \) with an overlap domain \( D \), then for any \( \delta \in D \):

\[
\lim_{\epsilon \to 0^+} \frac{f(x; \epsilon) - g(x; \epsilon)}{\eta(\epsilon)} = 0, \tag{6}
\]

uniformly. We then say \( f(x; \epsilon) \sim g(x; \epsilon) \) uniformly to order \( \eta(\epsilon) \) in \( D \) as \( \epsilon \to 0 \).

In practice, suppose that \( f(x; \epsilon) \) is the outer asymptotic expansion of \( y(x; \epsilon) \) valid on the extended domain to the right of \( \delta_1(\epsilon) \), and \( g(x; \epsilon) \) the inner asymptotic expansion valid on the extended domain to the left of \( \delta_2(\epsilon) \), with the overlap domain of validity \( D \), see Figure 1. The intermediate limit (6) is obtained by evaluating \( \frac{f(x; \epsilon) - g(x; \epsilon)}{\eta(\epsilon)} \) as \( \epsilon \to 0 \) along the curve \( x = \delta(\epsilon) \), where \( \delta(\epsilon) \) is any function \( \delta_1(\epsilon) \ll \delta(\epsilon) \ll \delta_2(\epsilon) \) as \( \epsilon \to 0 \).

![Figure 1: Intermediate matching. The region between the functions \( \delta_1(\epsilon) \) and \( \delta_2(\epsilon) \) is overlap domain \( D \). The function \( \delta(\epsilon) \) is a generic function in \( D \).](image)

In the case of the function (1), \( y_{\text{out}}(x) \) is uniform on any interval with left end point much larger than \( \delta_1(\epsilon) = \epsilon \) as \( \epsilon \to 0^+ \), while the inner expansion \( y_{\text{in}}(x) \) is uniform on any interval with right end point much smaller than \( \delta_2(\epsilon) = 1 \) as \( \epsilon \to 0^+ \). Then for any function \( \delta(\epsilon) \) satisfying the requirement \( \epsilon \ll \delta(\epsilon) \ll 1 \) as \( \epsilon \to 0^+ \), for example, \( \delta(\epsilon) = \epsilon^a \) with \( 0 < a < 1 \), the intermediate limit (6) to order \( \eta(\epsilon) = 1 \) reads

\[
\lim_{\epsilon \to 0^+} \frac{y_{\text{out}}(x) - y_{\text{in}}(x)}{\eta(\epsilon)} = \lim_{\epsilon \to 0^+} \left[ \zeta_\delta \delta(\epsilon) - e^{-\zeta_\delta \delta(\epsilon)/\epsilon} \right] = 0,
\]
so that $y_{in}(x) \sim y_{out}(x)$ uniformly to order $O(1)$ as $\epsilon \to 0^+$ in the intermediate region $D : \epsilon \ll x \ll 1$.

Typically the outer and/or inner expansions may contain constants that remain undermined even after all conditions are imposed. For example, when constructing approximate solutions to differential equations. When this occurs, the constants are fixed by the requirement that the approximations match in the intermediate region because intermediate matching can only occur if they assume certain values. We shall come back to this point.

While simple in principle, intermediate matching can be difficult to be implemented in practice. The reason is that it is not always easy to determine the extended domain of validity of the asymptotic expansion. Thus, very often intermediate matching is performed using some simpler method, verifying a posteriori the consistency of the results. For example, if an overlap domain $D$ exists then $x \to 0$ as $\epsilon \to 0$ at the left boundary $\delta_1(\epsilon)$ of the extended domain of validity of the outer asymptotic expansion. Similarly $\xi \to \infty$ as $\epsilon \to 0$ at the right boundary $\delta_2(\epsilon)$ of the extended domain of validity of inner asymptotic expansion. Thus intermediate matching can be done by matching the $x \ll 1$ expansion of $f(x; \epsilon)$ with the $\xi \gg 1$ expansion of $g(x; \epsilon)$ at fixed $\epsilon$ to order $\eta(\epsilon)$. A more accurate method is to check if intermediate matching works for some $\delta(\epsilon)$ in $D$. For instance, $\delta(\epsilon) = \epsilon^a$ with $0 < a < 1$ as in the example.

These methods provide a simple procedure to perform asymptotic matching, whose validity can be verified a posteriori. However, one should be aware that they might fail for problems more complicated than the ones we shall consider.

So far, so good. We have matched the expansions. But how do we construct an uniform approximation in the whole domain of interest? It is evident that if the outer asymptotic expansion $y_{out}(x; \epsilon)$ and the inner asymptotic expansion $y_{in}(x; \epsilon)$ of $y(x; \epsilon)$ match to order $\eta(\epsilon)$ in the overlap domain $D$, then in $D$ they must assume the same functional form $y_{match}(x; \epsilon)$, i.e.,

$$y_{in}(x; \epsilon) \sim y_{out}(x; \epsilon) \sim y_{match}(x; \epsilon), \quad \text{as } \epsilon \to 0,$$

to order $\eta(\epsilon)$ uniformly in $D$. Then

$$y_{unif}(x; \epsilon) = y_{in}(x; \epsilon) + y_{out}(x; \epsilon) - y_{match}(x; \epsilon),$$

is an uniform asymptotic expansion of $y(x; \epsilon)$ to order $\eta(\epsilon)$ as $\epsilon \to 0$ in the whole domain. This can be easily seen by writing

$$y_{out}(x; \epsilon) = y_{match}(x; \epsilon) + R_{out}(x; \epsilon),$$
$$y_{in}(x; \epsilon) = y_{match}(x; \epsilon) + R_{in}(x; \epsilon),$$

where

$$\lim_{\epsilon \to 0} \frac{R_{out}(x; \epsilon)}{\eta(\epsilon)} = 0, \quad \text{in the extended domain of validity of } y_{in}(x; \epsilon),$$

$$\lim_{\epsilon \to 0} \frac{R_{in}(x; \epsilon)}{\eta(\epsilon)} = 0, \quad \text{in the extended domain of validity of } y_{out}(x; \epsilon).$$
In the example $y_{\text{match}}(x) = 0$ to order $O(1)$ because
\[
\lim_{\delta \to 0^+} \frac{y_{\text{in}}(x)}{\eta(\epsilon)} = \lim_{\delta \to 0^+} \frac{y_{\text{out}}(x)}{\eta(\epsilon)} = 0,
\]
uniformly in the overlap domain $D : \epsilon \ll x \ll 1$ with $\eta(\epsilon) = 1$. Then
\[
y_{\text{unif}}(x) = e^{-x/\epsilon} + x - \frac{0}{y_{\text{match}}} = e^{-x/\epsilon} + x.
\]
gives an uniform asymptotic expansion to $O(1)$ of the function $y(x; \epsilon)$ in (1) as $\epsilon \to 0^+$ in the whole interval $0 \leq x \leq 1$; indeed for $\eta(\epsilon) = 1$
\[
\lim_{\epsilon \to 0^+} \frac{y(x; \epsilon) - y_{\text{unif}}(x; \epsilon)}{\eta(\epsilon)} = \lim_{\epsilon \to 0^+} \frac{\epsilon}{\eta(\epsilon)} = 0
\]
uniformly for all $x \in [0, 1]$.

\subsection*{1.2. Matched Asymptotic Expansion in practice.}

Generic layer-type problems have solutions $y(x; \epsilon)$ that change rapidly in a narrow layer of thickness $\delta$ somewhere in the domain of variation of $x$ when the parameter $\epsilon$ gets small.

Generally speaking, there can be more than one layer located at different point $x$, a viscous fluid flowing in a channel is an example, however these must be well separated. Only in this case the inner and outer regions of each layer can be clearly defined and the method of matched asymptotic expansion be applied. A problem with two or more well separated layers can always be reduced to single-layer problems that can be studied independently one from the others. The different single-layer approximations are then matched to construct an uniform approximation to the original problem valid in the whole region of interest.

The general scheme to construct an uniform approximation to the solution of a layer-type problem using matched asymptotic expansions is the following. One first separates the region of interest into the outer, away from the layer, and the inner, inside the layer, regions. We assume that the layer is located at $x = 0$. In each region the solution $y(x; \epsilon)$ behaves differently as $\epsilon \to 0$. In the outer region $y(x; \epsilon)$ is slowly varying, i.e., it varies of $O(1)$ on intervals of $O(1)$ as $\epsilon \to 0$. The derivatives with respect to $x$ remain finite and give negligible contributions as $\epsilon \to 0$ if multiplied by vanishingly small terms. In the inner region $y(x; \epsilon)$ is rapidly varying: it varies of $O(1)$ on intervals $x = O(\delta)$ where $\delta = \delta(\epsilon)$, the thickness of the layer, vanishes as $\epsilon \to 0$. Hence, the derivatives with respect to $x$ becomes very large as $\epsilon \to 0$ and may give appreciable contributions even if multiplied by vanishingly small coefficients. By constrast, since $x = O(\delta)$ in the inner region, the form of $x$-dependent coefficients can be simplified when $\epsilon \to 0$.

The successive step uses Poincaré type expansions to construct two uniform asymptotic expansions of the solution as $\epsilon \to 0$: $y_{\text{out}}(x; \epsilon)$ and $y_{\text{in}}(x; \epsilon)$, one uniform in the outer and the other in the inner region. These expansions are also called the outer solution and the inner solution, respectively. The outer and
inner asymptotic expansions are matched in the common region of validity using intermediate matching to obtain the uniform asymptotic expansion $y_{\text{uni}}(x; \epsilon)$ of the solution as $\epsilon \to 0$ valid in the whole region of interest.

In practice, in the outer region we construct the asymptotic expansion of Poincaré type with gauge functions $\varphi_n(\epsilon)$,

$$y_{\text{out}}(x; \epsilon) \sim \sum_{n=0}^{\infty} y_n(x) \varphi_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

where

$$y_N(x) = \lim_{\epsilon \to 0} \frac{y(x; \epsilon) - \sum_{n=0}^{N-1} y_n(x) \varphi_n(\epsilon)}{\varphi_N(\epsilon)}.$$

The limit $\epsilon \to 0$ is taken at fixed $x$ (outer limit). A typical choice for the gauge functions is $\varphi_n(\epsilon) = \epsilon^n$. In general the gauge functions depend on the problem and, moreover, the outer and inner regions may require different gauge functions. The asymptotic expansion $y_{\text{out}}(x; \epsilon)$ is uniform in the outer region, but becomes non uniform as $x$ gets too close to the layer. As a rule of thumb this occurs when two successive terms of (7) are of the same order, which naively occurs when $x/\delta = O(1)$ as $\epsilon \to 0$.

To construct an uniform asymptotic expansion of $y(x; \epsilon)$ as $\epsilon \to 0$ in the inner region, we first introduce the rescaled or inner variable $\xi = x/\delta$ and rewrite $y(x; \epsilon) \to y(\xi; \epsilon) = y(\delta \xi; \epsilon)$. For finite $\epsilon$ this is just a change of variable, hence $y(x; \epsilon)$ and $y(\xi; \epsilon)$ are completely equivalent. However, in the limit $\epsilon \to 0$ if $\xi$ is held fixed we will never leave the layer, even if the width $\delta \to 0$. Thus, if we construct the asymptotic expansion of Poincaré type

$$y_{\text{in}}(\xi; \epsilon) \sim \sum_{n=0}^{\infty} Y_n(\xi) \varphi_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

where

$$Y_N(\xi) = \lim_{\epsilon \to 0} \frac{y(\xi; \epsilon) - \sum_{n=0}^{N-1} Y_n(\xi) \varphi_n(\epsilon)}{\varphi_N(\epsilon)}$$

$$= \lim_{\epsilon \to 0} \frac{y(x; \epsilon) - \sum_{n=0}^{N-1} Y_n(x) \varphi_n(\epsilon)}{\varphi_N(\epsilon)},$$

with the limit $\epsilon \to 0$ taken at fixed $\xi = x/\delta$ (inner limit), we obtain an asymptotic expansion of $y(x; \epsilon)$ inside the layer. To distinguish between the outer and inner expansions we use lowercase letters for coefficients of the outer expansion and uppercase letters for those of the inner expansion. We also assumed that we can use the same gauge functions for the inner and outer expansion. The
asymptotic expansion \( y_{\text{in}}(x/\delta; \epsilon) \) is uniform inside the layer, however it becomes non uniform as we move too far from the layer.

If the domains of validity of the outer and inner uniform asymptotic expansions \( y_{\text{out}}(x; \epsilon) \) and \( y_{\text{in}}(\xi; \epsilon) \) have a common domain \( D \) of validity, then they can be matched in \( D \) by performing the intermediate limit (6). In most cases the intermediate limit can be done by matching to order \( \eta(\epsilon) \) the expansion of \( y_{\text{out}}(x; \epsilon) \) for \( x \ll 1 \) with the expansion of \( y_{\text{in}}(\xi; \epsilon) \) for \( \xi = x/\delta \gg 1 \) at fixed \( \epsilon \). That is, if we define the inner limit of the outer expansion as:

\[
y_{\text{out}}(x; \epsilon)|_{\text{in}} = \text{expansion of } y_{\text{out}}(x; \epsilon) \text{ for } x \ll 1 \text{ and fixed } \epsilon,
\]

and the outer limit of the inner expansion as:

\[
y_{\text{in}}(\xi; \epsilon)|_{\text{out}} = \text{expansion of } y_{\text{in}}(\xi; \epsilon) \text{ for } \xi \gg 1 \text{ and fixed } \epsilon,
\]

then the intermediate limit implies that

\[
y_{\text{in}}(\xi; \epsilon)|_{\text{out}} \sim y_{\text{out}}(x; \epsilon)|_{\text{in}} \sim y_{\text{match}}(x; \epsilon) \quad \text{as } \epsilon \to 0,
\]

to order \( \eta(\epsilon) \) uniformly in \( D \). As a rule of thumb, to evaluate the relative order of the different terms of the expansions in the overlap domain as \( \epsilon \to 0 \) it might be useful to think that \( x = O(\delta) \) and \( \xi = O(1/\delta) \). This assumption is not correct in \( D \), nevertheless it usually gives the correct relative orders as \( \epsilon \to 0 \).

To establish more precisely the order of the various terms and the extended domain of validity of the expansions in the \( x\epsilon \)-plane we can set \( x = \delta(\epsilon)\zeta \delta \) for some fixed \( \zeta \delta \neq 0 \) and some function \( \delta(\epsilon) \) that remains bounded as \( \epsilon \to 0 \). If \( \delta = O(1) \) then \( x \) is fixed as in the outer limit, while if \( \delta \) equals the width of the layer then \( x \) tends to zero as \( \epsilon \to 0 \) at the same rate as the thickness of the layer as in the inner limit. In order for the intermediate limit to be valid the matching (9) must hold for any function within these two extrema, but not equal to any of them.

We stress that the success (or failure) of the method of matched asymptotic expansions relies on the existence of the overlap domain \( D: \delta_1 \ll x \ll \delta_2 \), with \( \delta_2/\delta_1 \gg 1 \), as \( \epsilon \to 0 \). As already noted, the width of overlap domain usually shrinks to 0 in the variable \( x \) and diverges in the inner variable \( \xi \). The width of the overlap domain generally depends on the order \( \eta(\epsilon) \) of the approximation and typically decreases as this increases. Thus matching might fail to some order where the requirement \( \delta_2/\delta_1 \gg 1 \) as \( \epsilon \to 0 \) is no longer satisfied.

If an overlap domain \( D \) exists, and the outer and inner uniform asymptotic expansions can be matched in \( D \) to order \( \eta(\epsilon) \) as \( \epsilon \to 0 \), then,

\[
y_{\text{unif}}(x; \epsilon) = y_{\text{out}}(x; \epsilon) + y_{\text{in}}(x/\delta; \epsilon) - y_{\text{match}}(x; \epsilon),
\]

where \( \delta \) is the thickness of the layer, gives an uniform asymptotic expansion of \( y(x; \epsilon) \) to order \( \eta(\epsilon) \) as \( \epsilon \to 0 \) on the whole domain of interest.

The procedure is valid for any layer-type problem. However, for simplicity, we shall first consider boundary-layer problems where the layer is at one edge of region of interest.
Example 1.1. Consider the second order differential equation

\[ \epsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0, \quad 0 \leq x \leq 1 \]  
\[ y(0; \epsilon) = 0, \quad y(1; \epsilon) = 1, \]  
(10)

where \( \epsilon \) is a small positive parameter. This equation can be solved exactly and the general solution is

\[ y(x; \epsilon) = Ae^{\lambda_+ x} + Be^{\lambda_- x}, \]

where

\[ \lambda_\pm = \frac{1}{2\epsilon} \left[-1 \pm \sqrt{1 - 4\epsilon}\right]. \]  
(11)

Imposing the boundary conditions at \( x = 0 \) and \( x = 1 \) we have

\[ y(x; \epsilon) = \frac{e^{\lambda_+ x} - e^{\lambda_- x}}{e^{\lambda_+} - e^{\lambda_-}}. \]  
(12)

When \( \epsilon \to 0^+ \) the solution changes rapidly close to \( x = 0 \) because in this limit \( \lambda_+ = O(1) \) while \( \lambda_- = O(1/\epsilon) \). Thus at \( x = 0 \) there is a boundary layer of width \( \delta = O(\epsilon) \) as \( \epsilon \to 0^+ \).

Let us construct an uniform approximation to the solution of the differential equation (10) as \( \epsilon \to 0^+ \) using the method of matched asymptotic expansions. We shall present the calculation in some detail, at the price of being pedantic, to clarify the possible subtleties of the method.

- **Outer expansion**

  In the outer region, away from \( x = 0 \), we construct the Poincaré type asymptotic expansion (7) with gauge functions \( \varphi_n(\epsilon) = \epsilon^n \):

  \[ y_{\text{out}}(x; \epsilon) \sim y_0(x) + \epsilon y_1(x) + O(\epsilon^2), \quad \text{as } \epsilon \to 0^+. \]  
(13)

The functions \( y_n(x) \) satisfy the boundary condition

\[ y_0(1) = y(1; \epsilon) = 1, \quad \text{and} \quad y_n(1) = 0 \text{ for } n \geq 1. \]  
(14)

This is a useful choice. In principle, we could impose \( y_{\text{out}}(1; \epsilon) = y(1; \epsilon) = 1 \). Clearly a choice where the boundary value \( y(1; \epsilon) \) is not imposed only to \( y_0(x) \) leads to functions \( y_n(x) \) with parameters that depend on (powers of) \( \epsilon \). Thus, by modifying the definition of \( y_n(x) \), it is always possible to transform the asymptotic expansion into the form (13) with \( y_n(x) \) satisfying the boundary condition (14).

Inserting the expansion (13) into the differential equation (10) and equating the coefficients of equal power of \( \epsilon \), we find to the leading order

\[ \frac{dy_0}{dx} + y_0 = 0, \]  
(15)
whose general solution is
\[ y_0(x) = a_0 e^{-x}. \]
The arbitrary constant \( a_0 \) is fixed from the condition boundary condition (14); imposing \( y_0(1) = 1 \) gives \( a_0 = e \), and hence
\[ y_0(x) = e^{1-x}. \]
(16)

The outer expansion \( y_{\text{out}}(x; \epsilon) \sim y_0(x) \) gives an uniform asymptotic expansion of \( y(x; \epsilon) \) to \( O(1) \) as \( \epsilon \to 0^+ \) in any interval \( 0 < x_0 \leq x \leq 1 \), where \( x_0 \) is a positive constant independent of \( \epsilon \).

Note that \( y_0(x) \) cannot satisfy the boundary at \( x = 0 \); imposing \( y(0; \epsilon) = 0 \) gives \( a_0 = 0 \) leading to the trivial solution \( y_0(x) = 0 \), hence \( x = 0 \) cannot lie in the outer region. The failure of the outer solution to satisfy a boundary condition is a clear indication of the presence of a boundary layer.

Here we have a subtle point. We have seen that in general it is not justified to differentiate termwise an asymptotic expansion because subdominant terms may become relevant. Here, nevertheless, we have just done it. The reason is that \( y(x; \epsilon) \) satisfies a differential equation relating \( y(x; \epsilon) \) to its derivatives. Thus, if \( y(x; \epsilon) \) has an asymptotic expansion, its derivatives must have an asymptotic expansion with respect the same gauge functions without any extra contribution from subdominant terms. This ensures that we can safely differentiate the asymptotic expansion term-by-term. Alternatively, since termwise integration is safer, we may think of constructing an asymptotic expansion for the highest derivative, here \( y''(x; \epsilon) \), and then the asymptotic expansion of \( y(x; \epsilon) \) by successive term-by-term integrations, transforming de facto the differential equation into an integral equation. Hence, in either case the asymptotic expansion of the derivatives appearing in the differential equation can be obtained by termwise differentiation of the asymptotic expansion of \( y(x; \epsilon) \).

- **Inner expansion**
  The boundary layer at \( x = 0 \) has thickness \( \delta = O(\epsilon) \) as \( \epsilon \to 0^+ \). Without loss of generality we can take \( \delta = \epsilon \). Introducing the inner variable \( \xi = x/\epsilon \), and using the chain rule
  \[ \frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} = \frac{1}{\epsilon} \frac{d}{d\xi}, \]
the differential equation (10) becomes
\[ \frac{d^2 y}{d\xi^2} + \frac{dy}{d\xi} + \epsilon y = 0. \]
(17)
where \( y(\xi; \epsilon) = y(\epsilon\xi; \epsilon) \). Following the procedure outlined above, we next construct an asymptotic expansion \( y_{\text{in}}(\xi; \epsilon) \) of \( y(\xi; \epsilon) \), solution of the differential equation (17), of the Poincarè type with gauge function \( \phi_n(\epsilon) = \epsilon^n \):
\[ y_{\text{in}}(\xi; \epsilon) \sim Y_0(\xi) + \epsilon Y_1(\xi) + O(\epsilon^2), \quad \text{as } \epsilon \to 0^+. \]  (18)
The inner solution must fulfill the boundary condition at $x = 0$, but not the one at $x = 1$; thus, the coefficients $Y_n(\xi)$ satisfy the condition
$$Y_0(0) = y(0; \epsilon) = 0, \quad \text{and} \quad Y_n(0) = 0 \quad \text{for} \quad n \geq 1.$$ (19)
Inserting the expansion (18) into the differential equation (17), we have:
$$\frac{d^2 Y_0}{d \xi^2} + \frac{d Y_0}{d \xi} = 0.$$ The solution to this equation that satisfies the condition $Y_0(0) = 0$ is
$$Y_0(x) = A_0 \left(1 - e^{-\xi}\right),$$ (20)
where $A_0$ is an arbitrary constant.

The leading term $Y_0(\xi)$ of the inner expansion provides an uniform asymptotic expansion of $y(x; \epsilon)$ to $O(1)$ as $\epsilon \to 0^+$ on any interval $0 \leq \xi \leq \xi_0 < 1/\epsilon$, where $\xi_0$ is a positive constant independent of $\epsilon$.

The presence of an undetermined constant in $Y_0(\xi)$ gives some freedom in the asymptotic matching with the outer solution. If the outer and inner expansions were both completely determined we would have no tunable parameters to impose the asymptotic matching.

- **Intermediate matching**

The unknown constant $A_0$ is determined by performing the intermediate limit (6) in the overlap domain $D$ of the extended domains of validity of the two expansions and imposing that the outer and inner expansions match asymptotically to order $O(1)$ in the intermediate region. Since we do not know the extended domains of validity of the two expansions, we shall use the procedure of matching the expansions $y_{\text{out}}(x; \epsilon)|_{\text{in}}$ and $y_{\text{in}}(\xi; \epsilon)|_{\text{out}}$ to $O(1)$, verifying a posteriori the existence of $D$.

The inner limit $x \ll 1$ of $y_{\text{out}}(x; \epsilon)$ to $O(1)$ is
$$y_{\text{out}}(x; \epsilon)|_{\text{in}} = \left[ e^{1-x} + O(\epsilon) \right]_{x \ll 1} \sim e + O(x) + O(\epsilon), \quad x \ll 1.$$ (21)
Clearly $x$ cannot be too small because $y_{\text{out}}(x; \epsilon)$ is not uniform inside the boundary layer at $x = 0$. Heuristically, the breakdown occurs when the term $O(x)$ coming from the expansion of $y_0(x)$ for $x \ll 1$ becomes comparable with the term $O(\epsilon)$ coming from the (neglected) $\epsilon y_1(x)$ term. Hence the expansion (21) is valid on the interval $I_1 : \epsilon \ll x \ll 1$ as $\epsilon \to 0^+$.

Similarly, the outer limit of $y_{\text{in}}(x; \epsilon)$ to $O(1)$ reads:
$$y_{\text{in}}(\xi; \epsilon)|_{\text{out}} = \left[ A_0 \left(1 - e^{-\xi}\right) + O(\epsilon) \right]_{\xi \gg 1} \sim A_0 + O(e^{-\xi}) + O(\epsilon), \quad \xi \gg 1.$$ (22)
However $\xi$ cannot be arbitrarily large because $y_{\text{in}}(\xi; \epsilon)$ is not uniform outside the boundary layer. Therefore the expansion (22) is valid on $I_0 : 1 \ll \xi \ll 1/\epsilon$.

The domains of validity $I_0$ and $I_1$ overlap on:
$$D = I_0 \cap I_1 : \epsilon \ll x \ll 1 \quad \text{as} \quad \epsilon \to 0^+.$$
Notice that the ratio of the endpoints of the intermediate region \( \delta_2/\delta_1 = 1/\epsilon \gg 1 \) as \( \epsilon \to 0^+ \) and the width of \( D \) diverges in the inner variable \( \xi = x/\epsilon \).

Using (21) and (22) the intermediate limit of \( y_{\text{out}}(x; \epsilon) \) and \( y_{\text{in}}(\xi; \epsilon) \) to order \( \eta(\epsilon) = 1 \) in \( D \) is:

\[
\lim_{\epsilon \to 0^+} \frac{y_{\text{out}}(x; \epsilon) - y_{\text{in}}(x/\epsilon; \epsilon)}{\eta(\epsilon)} = \lim_{\epsilon \to 0^+} \left[ \epsilon - A_0 + O(\zeta_3 \delta) + O(e^{-\zeta_3 \delta/\epsilon}) + O(\epsilon) \right] = \epsilon - A_0,
\]

where \( \zeta_3 \) is fixed and \( \epsilon \ll \delta \ll 1 \) as \( \epsilon \to 0^+ \); hence the two expansions match if, and only if, \( A_0 = e \).

Thus

\[
y_{\text{unif}}(x; \epsilon) \sim y_{\text{out}}(x; \epsilon) + y_{\text{in}}(x/\epsilon; \epsilon) - y_{\text{match}}(x; \epsilon)
\]

\[
\sim e^{1-x} + e(1 - e^{-x/\epsilon}) - e + O(\epsilon)
\]

\[
\sim e^{1-x} - e^{1-x/\epsilon} + O(\epsilon), \quad \text{as } \epsilon \to 0,
\]

gives an asymptotic expansion of the solution \( y(x; \epsilon) \) of the differential equation (10) as \( \epsilon \to 0^+ \) valid uniformly on the whole interval \( 0 \leq x \leq 1 \).

Since we know the exact solution we can check the correctness of this conclusion. From (11) it follows that \( \lambda_+ \sim -1 + O(\epsilon) \) and \( \lambda_- \sim -1/\epsilon + O(1) \) as \( \epsilon \to 0^+ \), so that the exact solution (12) for small \( \epsilon \) reads

\[
y(x; \epsilon) = \frac{e^{-x+O(\epsilon x)} - e^{-x/\epsilon+O(x)}}{e^{-1+O(\epsilon)} - e^{-1/\epsilon+O(1)}} = \frac{e^{1-x+O(\epsilon x)} - e^{1-x/\epsilon+O(x)}}{e^{O(\epsilon)} - e^{-1/\epsilon+O(1)}}.
\]

It is straightforward to see that to order \( \eta(\epsilon) = 1 \),

\[
\lim_{\epsilon \to 0^+} \frac{y(x; \epsilon) - y_{\text{unif}}(x; \epsilon)}{\eta(\epsilon)} = 0,
\]

uniformly for \( x \in [0,1] \).

To construct an uniform asymptotic expansion of \( y(x; \epsilon) \) to \( O(\epsilon) \) as \( \epsilon \to 0^+ \) we have to include the terms \( y_1(x) \) and \( Y_1(\xi) \) of the outer and inner expansions. Substituting the outer expansion (13) into the differential equation (10) and equating the coefficients of equal powers of \( \epsilon \) up to \( O(\epsilon) \), we find that \( y_0 \) and \( y_1 \) satisfy the differential equations:

\[
O(1): \quad \frac{dy_0}{dx} + y_0 = 0,
\]

\[
O(\epsilon): \quad \frac{dy_1}{dx} + y_1 = -\frac{d^2y_0}{dx^2},
\]

supplemented by the boundary conditions (14): \( y_0(1) = 1 \) and \( y_1(1) = 0 \).

The system of equation (23) reveals the typical structure of perturbations methods with the leading term determined by a closed equation, and the next
terms dependent on previous ones. Inserting the solution (16) of the first equation into the second equation gives:

\[ \frac{dy_1}{dx} + y_1 = -e^{1-x}. \]

The general solution to this equation is:

\[ y_1(x) = a_1 e^{-x} - x e^{1-x}, \]

where \( a_1 \) is an arbitrary constant. The requirement \( y_1(1) = 0 \) fixes \( a_1 = e \), and:

\[ y_1(x) = (1 - x) e^{1-x}. \]

Thus, the outer expansion reads

\[ y_{\text{out}}(x; \epsilon) \sim e^{1-x} + \epsilon(1 - x) e^{1-x} + O(\epsilon^2), \quad \text{as } \epsilon \to 0^+. \tag{24} \]

To evaluate the next term in the inner expansion we substitute the expansion (18) into the differential equation (17) and, collecting the terms up to \( O(\epsilon) \), we get the set of differential equations:

\[
O(1) : \quad \frac{d^2 Y_0}{d\xi^2} + \frac{dY_0}{d\xi} = 0,
\]

\[
O(\epsilon) : \quad \frac{d^2 Y_1}{d\xi^2} + \frac{dY_1}{d\xi} = -Y_0.
\]

The functions \( Y_0(\xi) \) and \( Y_1(\xi) \) satisfy the boundary condition (19): \( Y_0(0) = 0 \) and \( Y_1(0) = 0 \). Replacing the solution (20) of the first equation into the second gives:

\[ \frac{d^2 Y_1}{d\xi^2} + \frac{dY_1}{d\xi} = -A_0 \left( 1 - e^{-\xi} \right). \]

The solution of this differential equation is

\[ Y_1(\xi) = A_1 + B_1 e^{-\xi} - A_0 \xi \left( 1 - e^{-\xi} \right), \]

with \( A_0, A_1 \) and \( B_0 \) arbitrary constant. Imposing the condition \( Y_1(0) = 0 \) we get \( B_1 = -A_1 \); thus,

\[ Y_1(\xi) = A_1 \left( 1 - e^{-\xi} \right) - A_0 \xi \left( 1 - e^{-\xi} \right), \]

and

\[ y_{\text{in}}(\xi; \epsilon) \sim A_0 \left( 1 - e^{-\xi} \right) + \epsilon \left( A_1 - A_0 \xi \right) \left( 1 - e^{-\xi} \right) + O(\epsilon^2), \quad \text{as } \epsilon \to 0^+. \tag{25} \]

The inner expansion \( y_{\text{in}}(\xi; \epsilon) \) depends upon the two arbitrary constants \( A_0 \) and \( A_1 \). This gives enough room to match the outer and inner expansions in the common domain of validity through an intermediate limit.
Taking the inner limit \( x \ll 1 \) to \( O(\epsilon) \) of the outer expansion (24) we get:\(^2\)

\[
y_{\text{in}}(x; \epsilon) \bigg|_{\epsilon} \sim e\left(1 - x + \epsilon + O(x^2) + O(\epsilon x) + O(\epsilon^2)\right), \quad x \ll 1,
\]

where the terms \( O(x^2) \) and \( O(\epsilon x) \) comes from the expansion of \( y_0(x) \) and \( \epsilon y_1(x) \), respectively, while \( O(\epsilon^2) \) is the unknown term from \( \epsilon^2 y_2(x) \). The consistency of the expansion \( \epsilon^2 y_2 \ll \epsilon y_1 \ll y_0 \) as \( \epsilon \to 0^+ \) requires \( x \gg \epsilon \) but \( x^2 \ll \epsilon \) as \( \epsilon \to 0^+ \); the expansion (26) is therefore valid only on the interval \( I_1 : \epsilon \ll x \ll \epsilon^{1/2} \) as \( \epsilon \to 0^+ \).

The outer limit \( \xi \gg 1 \) to \( O(\epsilon) \) of the inner expansion (25) gives:\(^3\)

\[
y_{\text{out}}(\xi; \epsilon) \bigg|_{\epsilon} \sim A_0 + \epsilon(A_1 - A_0\xi) + O(e^p\xi^q\exp(-\xi)) + O(\epsilon^2), \quad \xi \gg 1,
\]

where \( O(e^p\xi^q\exp(-\xi)) \) with \( p = 0, 1 \) and \( q = 0, 1 \) denotes the terms of the expansion of \( Y_0 + \epsilon Y_1 \) proportional to \( \exp(-\xi) \). These terms are subdominant because \( \exp(-\xi) \) is transcendentally small with respect to any power \( \xi^q \) as \( \xi \gg 1 \) and can be neglected. However \( \xi \) cannot be too large because consistency of the expansion, \( \epsilon Y_1 \ll Y_0 \) as \( \epsilon \to 0^+ \), requires \( \epsilon \xi \ll 1 \) as \( \epsilon \to 0^+ \); the expansion (27) is thus valid only on the interval \( I_0 : \epsilon \ll \xi \ll 1/\epsilon \) as \( \epsilon \to 0^+ \).

The outer and inner expansions to \( O(\epsilon) \) have the common domain of validity:

\[
D = I_0 \cap I_1 : \epsilon \ll x \ll \epsilon^{1/2}, \quad \text{as} \ \epsilon \to 0^+.
\]

The width of the intermediate region has reduced passing from \( \epsilon \ll x \ll 1 \) to \( \epsilon \ll x \ll \epsilon^{1/2} \). Nevertheless, the requirement \( \delta_2/\delta_1 = \epsilon^{-3/2} \gg 1 \) as \( \epsilon \to 0^+ \) is still fulfilled, and hence \( y_{\text{in}} \) and \( y_{\text{out}} \) can be matched asymptotically in \( D \).

The intermediate limit to order \( \eta(\epsilon) = \epsilon \) of (24) and (25) in \( D \) reads:

\[
\lim_{\epsilon \to 0^+} \frac{y_{\text{out}}(x; \epsilon) - y_{\text{in}}(x/\epsilon; \epsilon)}{\eta(\epsilon)} = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left[ (1 + \epsilon + \epsilon\delta - A_0 - \epsilon(A_1 - A_0\delta\epsilon)/(\epsilon)) \right] \\
= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left[ (1 - \epsilon\delta)(e - A_0) + \epsilon(e - A_1) \right],
\]

where \( \delta \) is fixed and \( \epsilon \ll \delta \ll \epsilon^{1/2} \) as \( \epsilon \to 0^+ \). Asymptotic matching requires \( A_0 = A_1 = e \). The value of \( A_0 \) is unchanged; this is a necessary condition for the method to work. Matching of \( y_{\text{out}}(x; \epsilon) \) and \( y_{\text{in}}(\xi; \epsilon) \) is not a trivial process because \( A_0 \) occurs in both \( O(1) \) and \( O(\epsilon) \) terms. Asymptotic matching works only if matching to a given order does not spoil the results from the matching at lower order.

Substituting \( A_0 = A_1 = e \) into the inner expansion we have,

\[
y_{\text{in}}(\xi; \epsilon) \sim e(1 - e^{-\xi}) + \epsilon e(1 - \xi)(1 - e^{-\xi}) + O(\epsilon^2), \quad \text{as} \ \epsilon \to 0^+.
\]

\(^2\)To identify the terms to retain in the expansion it may be useful to think that \( x = O(\epsilon) \), the width of the boundary layer. See discussion below equation (9).

\(^3\)To identify the terms to retain in the expansion it may be useful to think that \( \xi = O(1/\epsilon) \), the width of the boundary layer. See discussion below equation (9).
Combining together \( y_{\text{out}}(x; \epsilon) \), \( y_{\text{in}}(\xi; \epsilon) \) and \( y_{\text{match}}(x; \epsilon) \sim e(1+\epsilon-x) + O(\epsilon^2) \), we obtain:

\[
y_{\text{unif}}(x; \epsilon) \sim \left(1 - \epsilon x + \epsilon\right) e^{1-x} - \left(1 - \epsilon + \epsilon\right) e^{1-x} + O(\epsilon^2), \quad \text{as } \epsilon \to 0^+.
\] (28)

The asymptotic expansion (28) gives a uniform approximation to the solution of the differential equation (10) on the interval \( 0 \leq x \leq 1 \) that differs from the exact result by terms \( O(\epsilon^2) \) as \( \epsilon \to 0^+ \).

Note that \( y_{\text{unif}}(x=0; \epsilon) = O(\epsilon^2) \) and the boundary condition is satisfied to \( O(\epsilon) \). Similarly, at \( x=1 \) we have \( y_{\text{unif}}(x=1; \epsilon) = 1 + O(\epsilon) \), and the boundary condition is also satisfied to \( O(\epsilon) \) because \( e^{-1/\epsilon} \) is transcendentally small with respect to any power \( \epsilon^n \) as \( \epsilon \to 0^+ \).

We know the exact solution (12) thus we can check the approximation (28). From (11) we have

\[
\lambda_+ \sim -1 - \epsilon + O(\epsilon^2) \quad \text{and} \quad \lambda_- \sim -1/\epsilon + 1 + \epsilon + O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0^+,
\]

thus

\[
e^{\lambda_+ x} \sim \left[1 - \epsilon x + O(\epsilon^2)\right] e^{-x}, \quad \epsilon \to 0^+,
\]

and

\[
e^{\lambda_- x} \sim \left[1 + x + \epsilon x + O(\epsilon^2)\right] e^{-x/\epsilon} \\
\sim \left[1 + x + O(\epsilon^2)\right] e^{-x/\epsilon}, \quad \epsilon \to 0^+.
\]

The second line follows because \( e^{-x/\epsilon} = O(1) \) only if \( x = O(\epsilon) \) as \( \epsilon \to 0^+ \). Similarly, neglecting subdominant terms \( O(\epsilon^{-1/\epsilon}) \), we have

\[
\frac{1}{e^{\lambda_+} - e^{\lambda_-}} \sim e \left(1 + \epsilon + O(\epsilon^2)\right), \quad \text{as } \epsilon \to 0^+.
\]

Then as \( \epsilon \to 0^+ \) the exact result has the asymptotic expansion

\[
y(x; \epsilon) = \frac{e^{\lambda_+ x} - e^{\lambda_- x}}{e^{\lambda_+} - e^{\lambda_-}} \\
\sim \left[\left(1 - \epsilon x\right) e^{-x} - \left(1 + x\right) e^{-x/\epsilon} + O(\epsilon^2)\right] \times e \left[1 + \epsilon + O(\epsilon^2)\right] \\
\sim \left(1 - \epsilon x + \epsilon\right) e^{1-x} - \left(1 - x + \epsilon\right) e^{1-x/\epsilon} + O(\epsilon^2).
\]

Hence to order \( \eta(\epsilon) = \epsilon \)

\[
\lim_{\epsilon \to 0} \frac{y(x; \epsilon) - y_{\text{unif}}(x; \epsilon)}{\eta(\epsilon)} = 0,
\]

uniformly in \( x \in [0; 1] \).

2. Distinguished limit

In the Example 1.1 the unperturbed problem with \( \epsilon = 0 \) cannot satisfy the boundary condition \( y(x = 0; \epsilon) \) because of the boundary layer at \( x = 0 \). The failure to satisfy a boundary condition is an indication of the presence of a
boundary layer, or more generally of a layer somewhere. In general, however, such an information might not be available, thus we need a general procedure to identify the location of the layer and its width.

The idea is rather straightforward: we make a guess on the position of the layer and check the consistency of the hypothesis. That is, we postulate that the layer is at the position \( x_0 \) and has a width \( \delta \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). If our hypothesis leads to a trivial unperturbed problem or the outer and inner expansions cannot be matched, the proposition is rejected.

The procedure is plain, yet it contains a subtle point. To find the differential equation satisfied by the leading order term \( y_0(x) \) of \( y_{\text{out}}(x; \epsilon) \) we simply set \( \epsilon = 0 \) into the original differential equation. On the other hand, to get the differential equation for the leading order term \( Y_0(\xi) \) of \( y_{\text{in}}(\xi; \epsilon) \) we must first rewrite the differential equation in terms of the inner variable \( \xi = x/\delta \) and only then set \( \epsilon = 0 \). Unfortunately, this procedure cannot be carried on without knowing how the width \( \delta \) of the layer vanishes as \( \epsilon \rightarrow 0 \).

Luckily \( \delta \) can be determined by a suitable balance of the most divergent terms in the equation as \( \epsilon \rightarrow 0 \) through the limiting process called distinguished limit, as shown in the following example.

**Example 2.1.** Consider the differential equation

\[
\epsilon y'' + (x - 2) y' + y = 0, \quad 0 \leq x \leq 1
\]

\[
y(0; \epsilon) = 1, \quad y(1; \epsilon) = 1,
\]

where \( \epsilon \) is a small positive parameter. To simplify the notation here and in the following we shall denote the differentiation with respect to the independent variable by a “prime”. If differentiation is with respect to \( x \), as in this case, or with respect to \( \xi \) depends on the region, outer or inner, we are considering. When ambiguities might arise we shall use the extended notation.

The leading order term \( y_0(x) \) of the outer asymptotic expansion \( y_{\text{out}}(x; \epsilon) \) of the solution of the differential equation (29) satisfies the differential equation

\[
(x - 2) y' + y = 0,
\]

obtained just setting \( \epsilon = 0 \) into (29). The general solution to this equation is

\[
y_0(x) = \frac{a_0}{x - 2},
\]

where \( a_0 \) is an arbitrary constant whose value is determined from the boundary conditions. If we require \( y_0(0) = y(0; \epsilon) \) we get \( a_0 = -2y(0; \epsilon) = -2 \), if we instead require \( y_0(1) = y(1; \epsilon) \) we get \( a_0 = -y(1; \epsilon) = -1 \). We find two different values for \( a_0 \) because when \( \epsilon = 0 \) the differential equation becomes first order and cannot satisfies simultaneously the boundary conditions at the two ends of the interval. This is usually an indication, even if not a necessary nor a sufficient condition, for the presence of a singular layer somewhere in \([0; 1]\).

But, where is located the singular layer? And what is its width? The simpler hypothesis is that the layer is at the boundary. But which one, \( x = 0 \) or \( x = 1 \)?
In this case $y_0(x)$ does not help because it can satisfy the boundary condition on either side, so we have to check both.

Assuming that the singular layer is at the boundary $x = 0$ we introduce the inner variable $\xi = x/\delta$ where $\delta$ is the unknown width of the layer. All we know about $\delta$ is that it vanishes as $\epsilon \to 0^+$. Using the chain rule

$$\frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} = \frac{1}{\delta} \frac{d}{d\xi},$$

we change the variable $x \to \xi$ in (29) and get the differential equation,

$$\frac{\epsilon}{\delta^2} y'' + \left( \xi - \frac{2}{\delta} \right) y' + y = 0,$$

(31)

where $y(\xi; \epsilon) \equiv y(x = \delta\xi; \epsilon)$.

To find the differential equation satisfied by the leading order term $Y_0(\xi)$ of the asymptotic expansion $y_{\infty}(\xi; \epsilon)$ we must set $\epsilon = 0$ in (31). This can only be done making a dominance-balance hypothesis among the various terms as $\epsilon \to 0^+$; that is, an hypothesis on the order of magnitude of the different terms of the equation when $\delta \to 0$ as $\epsilon \to 0^+$. In the inner region we want to maintain the second order derivative, otherwise we get the first order differential equation of the outer region. Then there are only two possibilities: either $\epsilon/\delta^2 \gg 1/\delta$ or $\epsilon/\delta^2 \sim 1/\delta$ as $\epsilon \to 0^+$. Strictly speaking we should write $\epsilon/\delta^2 = O(1/\delta)$ as $\epsilon \to 0^+$ in strict sense. However, since for any non-null constant $a$ the rescaling $\delta \to a\delta$ does not change the relation, we can always chose $\delta$ so that $\epsilon/\delta^2 \sim 1/\delta$ as $\epsilon \to 0^+$.

The hypotesys $\epsilon/\delta^2 \sim 1$ as $\epsilon \to 0^+$ is not consistent because it leads to $\delta \sim \epsilon^{1/2}$ and the second order derivative disappears in the limit $\epsilon \to 0^+$ because $\epsilon/\delta^2 \ll 1/\delta$.

The hypotesys $\epsilon/\delta^2 \gg 1/\delta$, i.e., $\delta \ll \epsilon$, as $\epsilon \to 0^+$ implies that the most divergent term in (31) is the first one and the differential equation reduces to $Y_0'' = 0$ as $\epsilon \to 0^+$. Thus, $Y_0(\xi) = A_0\xi + B_0$, where $A_0$ and $B_0$ are constants.

The outer solution $y_0(x)$ has the finite inner limit $y_0(x) \sim -a_0/2 + O(x) = y(1; \epsilon)/2 + O(x)$ as $x \ll 1$, thus $A_0 = 0$ otherwise $Y_0(\xi)$ would diverge as $\xi \gg 1$ precluding the matching. This leaves only the free parameter $B_0$ and matching becomes problematic because there are not tuneable parameters; the condition $Y_0(0) = y(0; \epsilon)$ gives $B_0 = y(0; \epsilon)$ and the matching is not generic. It can be attained only for the particular boundary conditions $y(0; \epsilon) = y(1; \epsilon)/2$. Hence, the assumption $\delta \ll \epsilon$ as $\epsilon \to 0^+$ is rejected.

The proposition $\delta \ll \epsilon$ [as $\epsilon \to 0^+$] does not fix the order of $\delta$ but only that $\delta = o(\epsilon)$ as $\epsilon \to 0^+$. Statements of this type are called undistinguished limits because any choice $\delta = o(\epsilon)$ as $\epsilon \to 0^+$ satisfies the hypothesis. They are not useful to find scaling of the width of the singular layer.

On the other hand, the assertion $\epsilon/\delta^2 \sim 1/\delta$ as $\epsilon \to 0^+$ implies that $\delta$ must vanish as $\delta \to \epsilon$ in the limit of vanishing $\epsilon$. Statements of this type, which identify the order of $\delta$ as $\epsilon \to 0^+$, are called distinguished limit. Only distinguished limits lead to nontrivial layer structures because they involve a non-trivial dominance-balance among the most diverging terms of the differential equation; in our
example the balance between the first and the third term of the differential equation (31) as $\epsilon \to 0^+$.

Substituting $\delta = \epsilon$ in (31), and setting $\epsilon = 0$, we obtain the differential equation:

$$Y_0'' - 2Y_0' = 0,$$

with solution

$$Y_0(\xi) = A_0 e^{2\xi} + B_0.$$

The first term diverges as $\xi \gg 1$, thus matching with $y_{\text{out}}(x; \epsilon)$ can occur only for $A_0 = 0$. The condition $Y_0(0) = y(0; \epsilon)$ gives $B_0 = y(0; \epsilon)$; thus, also in this case the matching can be attained only for non generic boundary conditions.

The hypothesis of a boundary layer at $x = 0$ has not produced any useful result, and hence must be rejected.

To investigate the presence of a boundary layer at $x = 1$ we introduce the inner variable $\xi = (1 - x)/\delta$ where $\delta$ is the width of the layer. With this definition the inner variable vanishes at $x = 1$ and is positive for $x < 1$, inside the layer.

Using the the chain rule:

$$d \frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} = -\frac{1}{\delta} \frac{d}{d\xi},$$

from (29) we get the differential equation:

$$\frac{\epsilon}{\delta^2} y'' + \left(\frac{1}{\delta} + \xi\right) y' + y = 0,$$

where $y(\xi; \epsilon) \equiv y(x = 1 - \delta\xi; \epsilon)$.

The most divergent terms of the differential equation (32) as $\epsilon \to 0^+$ are the first two. The balancing of these two terms gives the distinguished limit $\epsilon/\delta^2 \sim 1/\delta$, i.e., $\delta \sim \epsilon$ as $\epsilon \to 0^+$. As in the previous case, the undistinguished limit gives a non generic matching.

Substituting $\delta = \epsilon$ into (32), and multiplying the resulting equation by $\epsilon$, we get

$$y'' + (1 + \epsilon\xi) y' + \epsilon y = 0.$$  \hspace{1cm} (33)

Setting $\epsilon = 0$ we obtain the differential equation

$$Y_0'' + Y_0' = 0.$$

for the first term $Y_0$ of the inner solution $y_{\text{in}}$. The solution to this equation with the boundary condition $Y_0(\xi = 0) = y(x = 1; \epsilon)$ is:

$$Y_0(\xi) = y(1; \epsilon) + A_0(e^{-\xi} - 1).$$  \hspace{1cm} (34)

At different with the previous case, the exponential term $e^{-\xi}$ does not diverge as $\xi \gg 1$ and $A_0$ need not to vanish. The value of $A_0$ is determined by the
intermediate matching of $y_{\text{in}}(\xi; \epsilon)$ and $y_{\text{out}}(x; \epsilon)$. The outer limit of $y_{\text{in}}(\xi; \epsilon) = Y_0(\xi) + O(\epsilon)$ to $O(1)$ is

$$y_{\text{in}}(\xi; \epsilon)|_{\text{out}} = \left[ y(1; \epsilon) + A_0(\epsilon^{-\xi} - 1) + O(\epsilon) \right]_{\xi \gg 1} \sim y(1; \epsilon) - A_0 + O(\epsilon^{-\xi}) + O(\epsilon),$$

and is valid uniformly on the interval $I_0 : 1 \ll \xi \ll 1/\epsilon$ as $\epsilon \to 0^+$. Since the boundary layer is at $x = 1$ the inner limit is taken for $x \to 1$; thus, the inner limit of $y_{\text{out}}(x; \epsilon) = y_0(x) + O(\epsilon)$ to $O(1)$ is

$$y_{\text{out}}(x; \epsilon)|_{\text{in}} = \left[ \frac{a_0}{x-2} + O(\epsilon) \right]_{1-x \ll 1} \sim 2y(0; \epsilon) + O(1-x) + O(\epsilon),$$

and is valid uniformly on $I_1 : \epsilon \ll 1-x \ll 1$ as $\epsilon \to 0^+$. We have explicitly replaced $a_0 = -2y(0; \epsilon)$ fixed by the requirement $y_0(0) = y(0; \epsilon)$.

The intervals $I_1$ and $I_0$ overlap and the expansions $y_{\text{in}}$ and $y_{\text{out}}$ can be matched asymptotically to $O(1)$ on $D = I_0 \cap I_1 : \epsilon \ll 1-x \ll 1$ as $\epsilon \to 0^+$. Matching of (35) and (36) gives

$$A_0 = y(1; \epsilon) - 2y(0; \epsilon),$$

thus

$$Y_0(\xi) = 2y(0; \epsilon) + \left[ y(1; \epsilon) - 2y(0; \epsilon) \right] \epsilon^{-\xi}.$$  

The expansions $y_{\text{in}}(\xi; \epsilon)$ and $y_{\text{out}}(x; \epsilon)$ can be matched asymptotically to $O(1)$ for any choice of boundary conditions, and hence the hypothesis that there is a boundary layer of width $\delta = \epsilon$ at $x = 1$ is consistent.

Notice that the coefficient of $\epsilon^{-\xi}$ vanishes for $y(1; \epsilon) = 2y(0; \epsilon)$ and $Y_0(\xi)$ becomes constant. This does not constitute a problem because it is a consequence of the boundary conditions, not of asymptotic matching.

With the boundary conditions $y(0; \epsilon) = y(1; \epsilon) = 1$ of the example, we get

$$y_{\text{out}}(x; \epsilon) = \frac{2}{2-x} + O(\epsilon),$$

$$y_{\text{in}}(\xi; \epsilon) = 2 - e^{-\xi} + O(\epsilon), \quad \text{as } \epsilon \to 0^+, \quad y_{\text{match}}(x; \epsilon) = 2 + O(\epsilon).$$

Therefore,

$$y_{\text{unif}}(x; \epsilon) \sim y_{\text{out}}(x; \epsilon) + y_{\text{in}}((1-x)/\epsilon; \epsilon) - y_{\text{match}}(x; \epsilon)$$

$$\sim \frac{2}{2-x} - e^{-(1-x)/\epsilon} + O(\epsilon), \quad \epsilon \to 0^+,$$

yields an uniform asymptotic expansion of the solution $y(x; \epsilon)$ of the differential equation (29) as $\epsilon \to 0^+$ valid uniformly to $O(1)$ in the whole interval $0 \leq x \leq 1$. To evaluate the term $O(\epsilon)$ of the expansion (38) we insert the Poincaré type asymptotic expansion

$$y_{\text{out}}(x; \epsilon) \sim y_0(x) + \epsilon y_1(x) + O(\epsilon^2), \quad \epsilon \to 0^+,$$
into the differential equation (29). Collecting the terms of equal power of \( \epsilon \) we get the set of differential equations:

\[
O(1) : (x - 2) y_0' + y_0 = 0, \\
O(\epsilon) : (x - 2) y_1' + y_1 = -y_0''.
\]

Inserting the solution of the first equation, given by (30) with \( a_0 = -2 \) to ensure that \( y_0(0) = y(0; \epsilon) = 1 \), into the second equation leads:

\[
(x - 2) y_1' + y_1 = \frac{4}{(x - 2)^3}.
\]

The general solution to this equation is

\[
y_1(x) = \frac{a_1}{x - 2} - \frac{2}{(x - 2)^3}.
\]

Requiring that \( y_1(0) = 0 \) gives \( a_1 = 1/2 \). Thus to \( O(\epsilon) \) we have

\[
yout(x; \epsilon) \sim 2 - x + \epsilon \left[ \frac{4}{(2 - x)^3} - \frac{1}{2 - x} \right] + O(\epsilon^2), \quad \epsilon \to 0^+.
\]

By comparing the first two terms for \( x \to 1^- \) it is straightforward to see that the expansion is uniform on \( 0 \leq x \leq x_0 < 1 \) with \( x_0 \) a positive constant strictly smaller that 1.

Similarly, in the inner region we consider the Poincaré type asymptotic expansion

\[
yin(\xi; \epsilon) \sim Y_0(\xi) + \epsilon Y_1(\xi) + O(\epsilon^2), \quad \epsilon \to 0^+,
\]

and the boundary conditions \( Y_0(0) = y(1; \epsilon) = 1 \) and \( Y_1(0) = 0 \). Inserting \( yin(\xi; \epsilon) \) into (33) and equating the terms of equal powers of \( \epsilon \), we get

\[
O(1) : Y_0'' + Y_0' = 0, \\
O(\epsilon) : Y_1'' + Y_1' = -\xi Y_0' - Y_0.
\]

Substituting from (34) \( Y_0(\xi) = 1 - A_0 + A_0 e^{-\xi} \), solution of the first equation with \( y(1; \epsilon) = 1 \), into the second gives the differential equation for \( Y_1 \):

\[
Y_1'' + Y_1' = A_0 - 1 - A_0 (1 - \xi) e^{-\xi}.
\]

The solution to this differential equation with the condition \( Y_1(0) = 0 \) is

\[
Y_1(\xi) = A_1 \left( e^{-\xi} - 1 \right) - \xi + A_0 \left( \xi - \frac{1}{2} \xi^2 e^{-\xi} \right)
\]

where \( A_1 \) is a constant. Collecting all terms we obtain

\[
yin(\xi; \epsilon) \sim 1 + A_0 \left( e^{-\xi} - 1 \right) + \epsilon \left[ A_1 \left( e^{-\xi} - 1 \right) - \xi + A_0 \left( \xi - \frac{1}{2} \xi^2 e^{-\xi} \right) \right] + O(\epsilon^2), \quad \epsilon \to 0^+.
\]
The expansion is uniform on the interval $0 \leq \xi \leq \xi_0 < 1/\epsilon$ with $\xi_0$ a positive constant.

The constants $A_0$ and $A_1$ are determined by the asymptotic matching of $y_{\text{out}}(x; \epsilon)$ and $y_{\text{in}}(\xi; \epsilon)$ in the common region of validity.

Taking the inner limit of (39) to $O(\epsilon)$ we get:

$$y_{\text{out}}(x; \epsilon)_{\text{in}} \sim 2 - 2(1-x) + O((1-x)^2)$$

$$+ \frac{3\epsilon}{2} + O(\epsilon(1-x)) + O(\epsilon^2), \quad 1-x \ll 1. \quad (42)$$

The expansion is valid on the interval $I_1 : \epsilon \ll 1-x \ll \epsilon^{1/2}$ where $(1-x)^2 \ll \epsilon$ but $1-x \gg \epsilon$ as $\epsilon \to 0^+$.

The outer limit of (41) to $O(\epsilon)$ gives:

$$y_{\text{in}}(\xi; \epsilon)_{\text{out}} \sim 1 - A_0 - \epsilon A_1 + (A_0 - 1)\epsilon \xi + O(\epsilon^p \xi^q e^{-\xi}) + O(\epsilon^2), \quad \xi \gg 1, \quad (43)$$

where $O(\epsilon^p \xi^q e^{-\xi})$ with $p = 0, 1$ and $q = 0, 1, 2$ indicates the terms of $Y_0 + \epsilon Y_1$ proportional to $e^{-\xi}$. These terms are subdominant for $\xi \gg 1$ and can be neglected. The requirement $\epsilon Y_1 \ll Y_0$ as $\epsilon \to 0^+$ implies that the expansion (43) is valid on the interval $I_0 : 1 \ll \xi \ll 1/\epsilon$ as $\epsilon \to 0^+$.

The intervals $I_0$ and $I_1$ overlap on the non-empty domain $D = I_0 \cap I_1 = \epsilon \ll 1-x \ll \epsilon^{1/2}$ as $\epsilon \to 0^+$. The width of the domain $D : \delta_1 \ll x \ll \delta_2$ where $y_{\text{in}}(\xi; \epsilon)$ and $y_{\text{out}}(x; \epsilon)$ are both valid to $O(\epsilon)$ is reduced with respect to the $O(1)$ result, yet $\delta_2/\delta_1 = \epsilon^{-1/2} \gg 1$ as $\epsilon \to 0^+$ and $y_{\text{in}}(\xi; \epsilon)$ and $y_{\text{out}}(x; \epsilon)$ can be matched to $O(\epsilon)$.

The asymptotic matching of $y_{\text{out}}(x; \epsilon)$ and $y_{\text{in}}(\xi; \epsilon)$ to $O(\epsilon)$ in $D$ requires that

$$y_{\text{out}}(1 - \delta \zeta; \epsilon)_{\text{in}} - y_{\text{out}}(\delta \zeta; \epsilon)_{\text{out}} = o(\epsilon), \quad \epsilon \to 0^+,$$

where $\zeta$ is constant and $\epsilon \ll \delta \ll \epsilon^{1/2}$. Inserting the expansions (42) and (43) we obtain the condition

$$2 + \frac{3\epsilon}{2} - 2\epsilon \zeta = 1 - A_0 - \epsilon A_1 + (A_0 - 1)\delta \zeta,$$

solved by $A_0 = -1$, as found from the matching to $O(1)$, and $A_1 = -3/2$. The inner solution to $O(\epsilon)$ then reads:

$$y_{\text{in}}(\xi; \epsilon) \sim 2 + \frac{3\epsilon}{2} - 2\epsilon \xi - \left[1 + \frac{3\epsilon}{2} - \frac{1}{2} \epsilon \xi^2\right] e^{-\xi} + O(\epsilon^2), \quad \epsilon \to 0^+. \quad (44)$$

Combining $y_{\text{out}}(x; \epsilon)$, $y_{\text{in}}(\xi; \epsilon)$ and $y_{\text{match}}(x; \epsilon)$, given respectively by (39), (44) and (42), gives

$$y_{\text{unif}}(x; \epsilon) \sim \frac{2}{2-x} + \frac{\epsilon}{2} \left[\frac{4}{(2-x)^3} - \frac{1}{(2-x)}\right]$$

$$- \left[1 + \frac{3\epsilon}{2} - \frac{(1-x)^2}{2\epsilon}\right] e^{-(1-x)/\epsilon} + O(\epsilon^2), \quad \epsilon \to 0^+. \quad (45)$$

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The asymptotic expansion (45) gives an uniform approximation to the solution \( y(x; \epsilon) \) of the differential equation (29) in the whole interval \( 0 \leq x \leq 1 \) that differs from \( y(x; \epsilon) \) by terms \( O(\epsilon^2) \) as \( \epsilon \to 0^+ \).

**Example 2.2.** Consider the differential equation (29) discussed in the previous example with the boundary condition to \( y(0; \epsilon) = 1 \) and \( y(1; \epsilon) = 2 \). This choice satisfy the identity \( y(1; \epsilon) = 2y(0; \epsilon) \) and, from (37), the inner solution to \( O(1) \) is constant:

\[
Y_0(\xi) = 2 + O(\epsilon), \quad \epsilon \to 0^+.
\]

The outer solution remains the same because the boundary condition \( y(0; \epsilon) = 1 \) is unchanged; thus the uniform approximation to \( O(1) \) is

\[
y_{\text{unif}}(x; \epsilon) \sim \frac{2}{2-x} + O(\epsilon), \quad \epsilon \to 0^+.
\]

The contribution from \( Y_0 \) is canceled out by \( y_{\text{match}} \).

At leading order in the outer region the differential equation (29) is first order and \( y_{\text{out}} \) remains unchanged to all orders in \( \epsilon \). The change of \( Y_0(\xi) \) propagates to the next orders; substituting \( Y_0 = 2 \) into the second differential equation (40), we have

\[
Y_1'' + Y_1' = -2
\]

that, imposing the boundary condition \( Y_1(0) = 0 \), gives:

\[
Y_1(\xi) = A_1(e^{-\xi} - 1) - 2\xi.
\]

Thus,

\[
y_{\text{in}}(\xi; \epsilon) \sim 2 + \epsilon [A_1 (e^{-\xi} - 1) - 2\xi] + O(\epsilon^2), \quad \epsilon \to 0^+.
\]

which is valid uniformly for \( 0 \leq \xi \leq \xi_0 < 1/\epsilon \) with \( \xi_0 \) positive and constant.

Taking the outer limit of \( y_{\text{in}} \) we have

\[
y_{\text{in}}(\xi; \epsilon)\big|_{\text{out}} \sim 2 - \epsilon A_1 - 2\epsilon \xi + O(\epsilon e^{-\xi}) + O(\epsilon^2), \quad \xi \gg 1, \quad (47)
\]

with domain of validity \( I_0 : 1 \ll \xi \ll 1/\epsilon \) as \( \epsilon \to 0^+ \).

The inner solution (46) can then be matched asymptotically with the outer solution (39) to \( O(\epsilon) \) as \( \epsilon \to 0^+ \) in the common domain of validity \( D = I_0 \cap I_1 = \epsilon \ll 1-x \ll e^{1/2} \). Using (47) and (42) it is straightforward to see that the asymptotic matching to \( O(\epsilon) \) in \( D \) requires \( A_1 = -3/2 \). Thus,

\[
y_{\text{unif}}(x; \epsilon) \sim \frac{2}{2-x} + \frac{\epsilon}{2} \left[ \frac{4}{(2-x)^3} - \frac{1}{(2-x)} \right] - \frac{3\epsilon}{2} e^{-(1-x)/\epsilon} + O(\epsilon^2), \quad \epsilon \to 0^+,
\]

gives an uniform asymptotic expansion of the solution \( y(x; \epsilon) \) of the differential equation (29) with boundary condition \( y(0; \epsilon) = 1 \) and \( y(1; \epsilon) = 2 \) valid uniformly in the interval \( 0 \leq x \leq 1 \) to \( O(\epsilon) \) as \( \epsilon \to 0^+ \).
Example 2.3. Consider the second order differential equation

$$
e y'' + a(x) y' + b(x) y = 0, \quad 0 \leq x \leq 1,$$

$$y(0; \epsilon) = A, \quad y(1; \epsilon) = B, \quad (48)$$

where $\epsilon$ is a small positive parameter and $a(x)$ and $b(x)$ generic bounded functions of $x$ on $0 \leq x \leq 1$. If $a(x) > 0$ for $x \in [0, 1]$, the solution $y(x; \epsilon)$ of the differential equation (48) has a boundary layer at $x = 0$ of width $\delta = \epsilon$ [as $\epsilon \to 0^+]$.

To show this we introduce the inner variable $\xi = x/\delta$, where $\delta$ is the width of the boundary, and rewrite the differential equation (48) as

$$\frac{\epsilon}{\delta^2} y'' + \frac{a(\delta \xi)}{\delta} y' + b(\delta \xi) y = 0, \quad (49)$$

where $y(\xi; \epsilon) = y(\delta \xi; \epsilon)$. Since $a(x)$ does not vanishes for $x \in [0, 1]$, the most diverging terms in (49) as $\epsilon \to 0^+$ are the first two, and hence, the consistent distinguished limit is $\epsilon/\delta^2 \sim 1/\delta$ as $\epsilon \to 0^+$. The distinguished limit $\epsilon/\delta^2 \sim 1$ [as $\epsilon \to 0^+$] leads indeed to the inconsistent result that the dominant term is the second one, while the distinguished limit $1/\delta \sim 1$ [as $\epsilon \to 0^+$] reproduces the outer limit.

The distinguished limit $\epsilon/\delta^2 \sim 1/\delta$ [as $\epsilon \to 0^+$] yields the width $\delta = \epsilon$. Thus, substituting $\delta = \epsilon$ into the differential equation (49)

$$y'' + a(\epsilon \xi) y' + \epsilon b(\epsilon \xi) y = 0,$$

and taking the limit $\epsilon \to 0^+$ at fixed $\xi$, gives the differential equation

$$Y_0'' + a(0) Y_0' = 0,$$

for the leading order term $Y_0$ of the inner expansion $y_{in}(\xi; \epsilon)$. The solution to this equation is

$$Y_0(\xi) = A_0 e^{-a(0) \xi} + B_0, \quad (50)$$

where $A_0$ and $B_0$ are two constants to be determined. The condition $Y_0(0) = y(0; \epsilon) = A$ gives the relation $A_0 + B_0 = A$. The second relation follows from the asymptotic matching of $Y_0(\xi)$ for $\xi \gg 1$ with the leading order term $y_{out}(x)$ of outer solution $y_{out}$ for $x \ll 1$. The assumption $a(x) > 0$ for $x \in [0, 1]$ ensures that $a(0) > 0$ and the first term in (50) is exponentially small as $\xi \gg 1$; thus, $A_0$ is not required to vanish.

The leading order term $y_0(x)$ of the outer solution $y_{out}(x; \epsilon)$ is given by the solution of the differential equation obtained by setting $\epsilon = 0$ into (48):

$$a(x) y_0' + b(x) y_0 = 0, \quad \Rightarrow \quad y_0(x) = a_0 e^{-\int^x ds b(s)/a(s)}, \quad (51)$$

where $a_0$ is a constant. The lower limit of integration is not specified because we can changed it by a suitable redefinition of $a_0$. Imposing the boundary condition $y_0(1) = y(1; \epsilon) = B$ removed the arbitrariness and gives:

$$y_0(x) = B e^{\int^x ds b(s)/a(s)}.$$
Since \( b(x) \) is bounded and \( a(x) \) does not vanish in \( 0 \leq x \leq 1 \), \( y_0(x) \) is well defined for all \( x \in [0, 1] \), and hence, the limit \( x \to 1 \) of \( y_0(x) \) is finite.

Matching the inner limit of the outer solution,

\[
y_{\text{out}}(x; \epsilon)|_{x \to 1} = B e^{\int_0^1 ds \frac{b(s)}{a(s)}} + O(x) + O(\epsilon), \quad x \to 1,
\]

with the outer limit of the inner solution,

\[
y_{\text{in}}(\xi; \epsilon)|_{\xi \to 1} = B_0 + O(e^{-\xi}) + O(\epsilon), \quad \xi \to 1
\]
to \( O(1) \) in the domain \( D : \epsilon \ll x \ll 1 \) as \( \epsilon \to 0^+ \), where both expansions are valid, gives

\[
B_0 = B e^{\int_0^1 ds \frac{b(s)}{a(s)}}.
\]

Combining \( y_{\text{out}} \), \( y_{\text{in}} \) and \( y_{\text{match}} \) we obtain the uniform asymptotic expansion,

\[
y_{\text{uni}}(x; \epsilon) \sim B e^{\int_0^1 ds \frac{b(s)}{a(s)}} + \left[ A - B e^{\int_0^1 ds \frac{b(s)}{a(s)}} \right] e^{-a(0)x/\epsilon} + O(\epsilon), \quad \epsilon \to 0^+,
\]

which provides an approximation to the solution of the differential equation \( (48) \) as \( \epsilon \to 0^+ \) valid uniformly to \( O(1) \) on the interval \( 0 \leq x \leq 1 \).

To arrive at the asymptotic expansion \( (52) \) the condition \( a(x) > 0 \) on the entire interval has played a central role; it has ensured that the term \( e^{-a(0)x} \) is subleading as \( \xi \to 1 \). In the case \( a(0) < 0 \), indeed, asymptotic matching can only occurs if \( A_0 = 0 \) leading to a non-generic matching.

Nevertheless, under the assumption that \( a(x) < 0 \) for \( 0 \leq x \leq 1 \) an analogous result can be derived. Inspection of \( (50) \) reveals that \( a(0) \) gives the rate of the exponential decay moving away on the right of the singular point \( x = 0 \). This suggests that if \( a(x) < 0 \) the boundary layer sits at \( x = 1 \) because in this case we move away on the left of the singular point. Indeed, introducing the inner variable \( \xi = (1 - x)/\delta \), the differential equation \( (48) \) becomes

\[
\frac{\epsilon}{\delta^2} y'' - \frac{a(1 - \delta\xi)}{\delta} y' + b(1 - \delta\xi) y = 0.
\]

The balancing the most diverging terms as \( \epsilon \to 0^+ \) leads the distinguished limit is \( \epsilon/\delta^2 \sim 1/\delta \) and gives \( \delta = \epsilon \). Substituting \( \delta = \epsilon \) into the differential equation \( (53) \), and setting \( \epsilon = 0 \) afterwards, we arrive at the differential equation whose solution gives the leading term \( Y_0 \) of the inner solution:

\[
Y_0'' - a(1) Y_0' = 0, \quad \Rightarrow \quad Y_0(\xi) = A_0 e^{a(1)\xi} + B_0.
\]

The constants \( A_0 \) and \( B_0 \) satisfy the constraint \( A_0 + B_0 = B \) which follows from the from the boundary condition \( Y_0(0) = y(1; \epsilon) \).

The exponential term \( e^{a(1)\xi} \) is exponentially small as \( \xi \to 1 \) because \( a(1) < 0 \), and \( Y_0(\xi) \) can be matched with the outer solution \( y_0(x) \) without any restriction on \( A_0 \). From \( (51) \), imposing the boundary condition \( y_0(0) = y(0; \epsilon) = A \), we have,

\[
y_0(x) = A e^{-\int_0^x ds \frac{b(s)}{a(s)}},
\]
so that, the asymptotic matching of $y_{\text{out}}(x; \epsilon)$ and $y_{\text{in}}(\xi; \epsilon)$ to $O(1)$ in the intermediate region $D : \epsilon \ll 1 - x \ll 1$ as $\epsilon \to 0^+$ leads to

$$B_0 = A e^{-\int_0^1 ds b(s)/a(s)}.$$  

Thus, combining all terms together, we have that

$$y_{\text{unif}}(x; \epsilon) \sim A e^{-\int_0^1 ds b(s)/a(s)}$$

$$+ \left[ B - A e^{-\int_0^1 ds b(s)/a(s)} \right] e^{a(1-x)/\epsilon} + O(\epsilon), \quad \epsilon \to 0^+.$$ 

gives an uniform approximation to the solution of the differential equation (48) as $\epsilon \to 0^+$ valid uniformly to $O(1)$ in the interval $0 \leq x \leq 1$.

3. Nonlinear Boundary Layer

In all examples discussed so far the gauge functions $\varphi_n(\epsilon)$ used in the Poincaré expansion were the same in the inner and outer regions; this is not a rule, however. If the gauge functions in the inner and outer regions differ matching $y_{\text{out}}(x; \epsilon)$ and $y_{\text{in}}(\xi; \epsilon)$ becomes more subtle.

A typical situation where gauge functions in the inner and outer expansions may differ is when the width $\delta$ of the boundary layer is nonlinear, as shown in the following example.

**Example 3.1.** Consider the second order differential equation:

$$\epsilon y'' + x^2 y' - y = 0, \quad 0 \leq x \leq 1,$$

$$y(0; \epsilon) = 1, \quad y(1; \epsilon) = 1,$$  

(54)

where $\epsilon$ is a small positive parameter. To construct an uniform approximation of the solution as $\epsilon \to 0^+$ in the interval $0 \leq x \leq 1$ the first step is to consider the Poincaré type expansion:

$$y_{\text{out}}(x; \epsilon) \sim y_0(x) + \epsilon y_1(x) + O(\epsilon^2), \quad \epsilon \to 0^+,$$  

(55)

in the outer region. The leading order term $y_0(x)$ of the expansion (55) is given by the solution of the differential equation obtained setting $\epsilon = 0$ into the original equation (54):

$$x^2 y'_0 - y_0 = 0.$$  

The solution to this equation is,

$$y_0(x) = a_0 e^{-1/x},$$  

(56)

where $a_0$ is a constant fixed by the boundary conditions. However, $y_0(x)$ cannot satisfy the boundary condition $y_0(0^+) = 1$ at $x = 0$, suggesting the possible presence of a boundary layer at $x = 0$. This conclusion agrees with the general results of Example 2.3 because $a(x) = x^2$ is positive for $0 < x \leq 1$ and we can
rule out a boundary layer at \( x = 1 \). The function \( a(x) \) vanishes at \( x = 0 \) it is, however, positive on any neighborhood \( 0 < x_0 \leq x \ll 1 \) where \( x_0 \) is an arbitrary constant. Thus, from Example 2.3 we may reasonably expect a boundary layer arbitrarily close to \( x = 0 \).

The condition \( y_0 (1) = 1 \) leads to \( a_0 = e \), and the leading order term of outer solution \( y_{\text{out}}(x) \) becomes:

\[
y_0(x) = e^{1 - 1/x}.
\]

To find the structure of the boundary layer at \( x = 0 \) we introduce the inner variable \( \xi = x/\delta \) and rewrite the differential equation (54) for the variable \( \xi \):

\[
\frac{\epsilon}{\delta^2} y'' + \delta \xi^2 y' - y = 0
\]

where \( y(\xi; \epsilon) \equiv y(\delta \xi; \epsilon) \). Balancing of the first and last term, the most diverging terms in the equation as \( \epsilon \to 0^+ \), gives the distinguished limit is \( \epsilon/\delta \sim 1 \); the width of the layer is thus \( \delta = \epsilon^{1/2} \). With the choice \( \delta = \epsilon^{1/2} \) the differential equation (56) becomes:

\[
y'' + \sqrt{\epsilon} \xi^2 y' - y = 0.
\]

Substituting the inner solution \( y_{\text{in}}(\xi; \epsilon) \sim Y_0(\xi) + o(1) \) as \( \epsilon \to 0^+ \) we obtain the differential equation

\[
Y_0'' - Y_0 = 0,
\]

with general solution

\[
Y_0(\xi) = A_0 e^{\xi} + B_0 e^{-\xi}.
\]

The boundary condition \( Y_0(\xi = 0) = 1 \) gives \( A_0 + B_0 = 1 \); however, the outer and inner expansions \( y_{\text{out}}(x; \epsilon) \) and \( y_{\text{in}}(\xi; \epsilon) \) cannot be matched unless \( A_0 = 0 \) because \( e^{\xi} \) is growing exponentially as \( \xi \gg 1 \). Thus \( B_0 = 1 \) and \( Y_0(\xi) \) reduces to:

\[
Y_0(\xi) = e^{-\xi}.
\]

There are no free parameter to tune the matching; for that reason for arbitrary boundary conditions \( y_{\text{out}}(x; \epsilon) \) and \( y_{\text{in}}(\xi; \epsilon) \) can be matched to \( O(1) \) only if in the intermediate region both \( e^{-1/x} \) and \( e^{-\xi} \) are transcendentally small with respect to any power of \( \epsilon \) as \( \epsilon \to 0^+ \). Taking \( x = \delta \xi, \) and \( \xi = \delta \zeta / \epsilon^{1/2} \), with \( \zeta \) finite and \( \delta \to 0 \) as \( \epsilon \to 0^+ \), the condition that \( e^{-1/x} \) and \( e^{-\xi} \) are transcendentally small with respect to any power \( \epsilon^n \) as \( \epsilon \to 0^+ \) leads to \( \sqrt{\epsilon} \ll \delta \ll 1 \). Thus in the intermediate region \( D : \epsilon^{1/2} \ll x \ll 1 \)

\[
y_{\text{out}}(x; \epsilon)|_{\text{in}} - y_{\text{in}}(\xi; \epsilon)|_{\text{out}} = o(1), \quad \epsilon \to 0^+,
\]

and \( y_{\text{in}} \) and \( y_{\text{out}} \) match to \( O(1) \) for arbitrary boundary conditions. Hence, since \( y_{\text{match}} = 0 \), we have that

\[
y_{\text{unit}}(x; \epsilon) \sim e^{1 - 1/x} + e^{-x/\sqrt{\epsilon}} + O(\epsilon) \quad \epsilon \to 0^+,
\]

\footnote{Alternatively, one can take \( x \sim \epsilon^a \), and transcendentally smallness leads to \( 0 < a < 1/2 \).}
is an asymptotic expansion of the solution of the differential equation (54) as \( \epsilon \to 0^+ \) valid uniformly to \( O(1) \) in the interval \( 0 \leq x \leq 1 \).

To compute the term \( O(\epsilon) \) of the asymptotic expansion we have to include the term \( y_1(x) \) of the outer expansion (55). Substituting \( y_{\text{out}}(x; \epsilon) \) into the differential equation (54), and collecting the terms proportional to \( \epsilon \), we find that \( y_1(x) \) is solution of the differential equation:

\[
x^2 y_1' - y_1 = -y_0'',
\]
which inserting \( y_0(x) = e^{1-1/x} \) becomes:

\[
x^2 y_1' - y_1 = \left( \frac{2}{x^3} - \frac{1}{x^4} \right) e^{1-1/x}.
\]

This equation must be solved with the boundary condition \( y_1(1) = 0 \). The solution to this equation is

\[
y_1(x) = a_1 e^{1-1/x} + \left( \frac{1}{5x^5} - \frac{1}{2x^4} \right) e^{1-1/x},
\]
and \( y_1(1) = 0 \) fixes \( a_1 = 3e/10 \). Thus \( y_{\text{out}}(x; \epsilon) \) reads:

\[
y_{\text{out}}(x; \epsilon) \sim \left( 1 + \frac{3\epsilon}{10} - \frac{\epsilon}{2x^4} + \frac{\epsilon}{5x^5} \right) e^{1-1/x} + O(\epsilon^2), \quad \epsilon \to 0^+.
\]

In the differential equation (57) expressed in terms of the inner variable \( \xi \) the small parameter is \( \epsilon^{1/2} \); and hence, the appropriate gauge functions in the inner region are \( \varphi_n(\epsilon) = \epsilon^{n/2} \). The Poincaré type expansion to \( O(\epsilon) \) in the inner region then reads:

\[
y_{\text{in}}(x; \epsilon) \sim Y_0(\xi) + \epsilon^{1/2} Y_1(\xi) + \epsilon Y_2(\xi) + O(\epsilon^{3/2}), \quad \epsilon \to 0^+.
\]

and we need both \( Y_1(\xi) \) and \( Y_2(\xi) \) to match \( y_{\text{out}}(x; \epsilon) \) to \( O(\epsilon) \). The boundary layer is relatively thick, \( \epsilon^{1/2} \gg \epsilon \) as \( \epsilon \to 0^+ \), and more terms of \( y_{\text{in}} \) are needed to match \( y_{\text{out}} \) in the intermediate region; for any new term in \( y_{\text{out}}(x; \epsilon) \) two terms in \( y_{\text{in}}(\xi; \epsilon) \) must be computed.

Substituting the expansion (59) into the differential equation (57), and collecting the coefficients of \( \epsilon^{1/2} \) and \( \epsilon \), we obtain the set of differential equations

\[
\begin{align*}
Y_1'' - Y_1 &= -\xi^2 Y_0', \\
Y_2'' - Y_2 &= -\xi^2 Y_1'.
\end{align*}
\]

which determine \( Y_1(\xi) \) and \( Y_2(\xi) \).

Inserting \( Y_0(\xi) = e^{-\xi} \) into the first equation gives

\[
Y_1'' - Y_1 = \xi^2 e^{-\xi},
\]
whose solution is:

\[
Y_1(\xi) = A_1 e^\xi + B_1 e^{-\xi} - \left( \frac{\xi}{4} + \frac{\xi^2}{4} + \frac{\xi^3}{6} \right) e^{-\xi}.
\]
The first term grows exponentially as $\xi \gg 1$ and asymptotic matching with the outer expansion is possible only if $A_1 = 0$. The boundary condition $Y_1(0) = 0$ further imposes $B_1 = 0$.

Replacing $Y_1(\xi)$ into the second equation (60) leads to:

$$Y''_2 - Y_2 = \left(\frac{\xi^2}{4} + \frac{\xi^3}{4} + \frac{\xi^4}{4} - \frac{\xi^5}{6}\right) e^{-\xi},$$

with general solution:

$$Y_2(\xi) = A_2 e^{\xi} + B_2 e^{-\xi} - \left(\frac{\xi}{32} + \frac{\xi^2}{32} + \frac{\xi^3}{48} - \frac{\xi^4}{96} - \frac{\xi^5}{60} - \frac{\xi^6}{72}\right) e^{-\xi}.$$

The constants are $A_2 = 0$, otherwise the first term diverges as $\xi \gg 1$ making matching impossible, and $B_2 = 0$ to satisfy the boundary condition $Y_2(0) = 0$. Hence $y_{in}(\xi; \epsilon)$ is:

$$y_{in}(\xi, \epsilon) \sim \left[1 - \epsilon^{1/2} \left(\frac{\xi}{4} + \frac{\xi^2}{4} + \frac{\xi^3}{6}\right) - \epsilon \left(\frac{\xi}{32} + \frac{\xi^2}{32} + \frac{\xi^3}{48} - \frac{\xi^4}{96} - \frac{\xi^5}{60} - \frac{\xi^6}{72}\right)\right] e^{-\xi} + O(\epsilon^{3/2}),$$

as $\epsilon \to 0^+$. In the intermediate region $D : \epsilon^{1/2} \ll x \ll 1$ both exponential $e^{-1/x}$ and $e^{-\xi}$ are transcendentally small with respect to any power of $\epsilon$ as $\epsilon \to 0^+$; thus,

$$y_{out}(x; \epsilon) - y_{in}(\xi; \epsilon) = o(\epsilon), \quad \epsilon \to 0^+,$$

uniformly in $D$.

Combining the two expansions leads to the asymptotic expansion

$$y_{unif}(x; \epsilon) \sim \left(1 + \frac{3\epsilon}{10} - \frac{\epsilon}{2x^4} + \frac{\epsilon}{5x^5}\right) e^{1-1/x} + \left(1 - \frac{x}{4} - \frac{x^2}{4e^{1/2}} - \frac{x^3}{6e} - \frac{\epsilon^{1/2} x}{32} - \frac{x^2}{32} - \frac{x^3}{48e^{1/2}} + \frac{x^4}{96e} + \frac{x^5}{60e^{1/2}} + \frac{x^6}{72e^2}\right) e^{-x/\sqrt{\epsilon}} + O(\epsilon^2), \quad \epsilon \to 0^+,$$

that gives an approximation to the solution of the differential equation (54) valid uniformly to $O(\epsilon)$ as $\epsilon \to 0^+$ in the interval $0 \leq x \leq 1$.

In this problem there is no interaction between the outer solution and the inner solution; in the intermediate region both $y_{out}(x; \epsilon)$ and $y_{in}(\xi; \epsilon)$ are transcendentally small with respect to $\epsilon^n$ as $\epsilon \to 0^+$, and the outer and inner expansions can be matched for arbitrary boundary conditions $y(0; \epsilon)$ and $y(1; \epsilon)$. 

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4. Boundary Layer at either extrema

The examples we have studied all exhibit a single singular layer located at one or the other of the two boundaries. Nothing, however, prevent the existence of two or more singular layers. If these are well separated, so that one can identify the inner and outer region of each layer, then the method of matched asymptotic expansions can be used.

Example 4.1. Consider the following second order differential equation

\[ \epsilon y'' - x^2 y' - y = 0, \quad 0 \leq x \leq 1 \]
\[ y(0; \epsilon) = 1, \quad y(1; \epsilon) = 1, \] (61)

where \( \epsilon \) is a small positive parameter. Setting \( \epsilon = 0 \) we find that the leading order term of \( y_{\text{out}}(x) \) is solution of the differential equation

\[ -x^2 y_0 - y_0 = 0, \quad \Rightarrow \quad y_0(x) = a_0 e^{1/x}. \]

Since \( y_0(x) \) diverges as \( x \to 0^+ \) the condition \( y_0(0) = y(0; \epsilon) = 1 \) cannot be imposed, and hence, there is a boundary layer at \( x = 0 \). Introducing the inner variable \( \xi = x/\delta \), the differential equation (61) becomes

\[ \frac{\epsilon}{\delta^2} y'' - \delta^2 y' - y = 0, \]

where \( y(\xi; \epsilon) \equiv y(x = \delta \xi; \epsilon) \). The consistent distinguished limit, obtained by balancing the first term with the last one, gives \( \delta \sim \epsilon^{1/2} \) as \( \epsilon \to 0^+ \). Substituting \( \delta = \epsilon^{1/2} \) the differential equation expressed in terms of the variable \( \xi = x/\epsilon^{1/2} \) becomes

\[ y'' - \sqrt{\epsilon} \xi^2 y' - y = 0. \]

The leading order term \( Y_0(\xi) \) of the expansion \( y_{\text{in},0}(\xi; \epsilon) \) inside the layer at \( x = 0 \) satisfies the differential equation:

\[ Y_0'' - Y_0 = 0, \]

with boundary condition \( Y_0(0) = y(0; \epsilon) = 1 \), hence:

\[ Y_0(\xi) = A_0 e^{\xi} + (1 - A_0) e^{-\xi}. \]

Matching with the outer expansion is possible only if \( A_0 = 0 \) because \( Y_0(\xi) \) diverges exponentially as \( \xi \gg 1 \), and there are no parameters left for the matching. However, \( Y_0(\xi) = O(e^{-\xi}) \) as \( x \ll 1 \) and hence the matching of \( y_{\text{in},0} \) with \( y_{\text{out}} \) to \( O(1) \) is still possible provided that \( y_0(x) \) is \( o(1) \) as \( \epsilon \to 0^+ \) in the intermediate domain. The function \( e^{1/x} \) is unbounded as \( x \to 0^+ \), and matching therefore imposes \( a_0 = 0 \). Consequently \( y_{\text{out}}(x; \epsilon) = O(\epsilon) \) as \( \epsilon \to 0^+ \), and \( y_{\text{in},0} \) and \( y_{\text{out}} \) match asymptotically to \( O(1) \) as \( \epsilon \to 0^+ \) in the intermediate region \( D : \epsilon^{1/2} \ll x \ll 1 \).

With \( a_0 = 0 \), however, \( y_0(x) \) does not satisfy the boundary condition \( y_0(1) = y(1; \epsilon) = 1 \); therefore, there must be a singular layer somewhere that prevents the
outer solution to reach the boundary $x = 1$, otherwise the condition $y(1; \epsilon) = 1$ cannot be fulfilled. Actually, by comparing with the Example 2.3 it follows that at $x = 1$ there is a boundary layer of width $\delta = \epsilon$ [as $\epsilon \to 0^+$] because the coefficient $a(x) = -x^2$ of the first derivative $y'$ is negative for $0 < x_0 \leq x \leq 1$.

Thus, introducing the inner variable $\xi = (1 - x)/\epsilon$, and using $(d/dx) = -(1/\epsilon)(d/d\xi)$, the differential equation (61) becomes,

$$y'' + (1 - 2\epsilon \zeta + \epsilon^2 \zeta^2)y' - \epsilon y = 0.$$ 

The leading order term $Y_0(\xi)$ of the expansion $y_{in,1}(\xi; \epsilon)$ inside the layer at $x = 1$ satisfies the differential equation

$$Y_0'' + Y_0' = 0.$$ 

The solution to this equation with the boundary condition $Y_0(0) = y(1; \epsilon) = 1$ is

$$Y_0(\xi) = A_0 + (1 - A_0)e^{-\xi}. $$ 

The value of the parameter $A_0$ is fixed by matching $y_{in,0}(\xi; \epsilon)$ and $y_{out}(x; \epsilon)$ to $O(1)$ as $\epsilon \to 0^+$ in the intermediate domain $D : \epsilon \ll 1 - x \ll 1$. Taking the $\delta$-limit with $x = 1 - \delta \zeta$, where $\zeta_\delta$ is constant and $\epsilon \ll \delta \ll 1$ as $\epsilon \to 0^+$, we have

$$\lim_{\epsilon \to 0^+} \left[ y_{out}(1 - \zeta_\delta \delta; \epsilon) - y_{in,1}(\zeta_\delta \delta/\epsilon; \epsilon) \right] = 0 - A_0,$$

so that matching requires $A_0 = 0$.

Combining $y_{out}(x; \epsilon)$ with the inner expansions $y_{in,0}(\xi; \epsilon)$ and $y_{in,1}(\xi; \epsilon)$ we obtain the uniform asymptotic expansion

$$y_{unif}(x; \epsilon) \sim y_{out}(x; \epsilon) + y_{in,0}(x/\sqrt{\epsilon}; \epsilon) - y_{match,0}(x; \epsilon) + y_{in,1}((1 - x)/\epsilon; \epsilon) - y_{match,1}(x; \epsilon)
\sim e^{-x/\sqrt{\epsilon}} + e^{-(1-x)/\epsilon} + O(\epsilon), \quad \epsilon \to 0^+. $$

The expansion (62) gives an approximation to the solution of the differential equation (61) valid uniformly to $O(1)$ as $\epsilon \to 0^+$ in the entire interval $0 \leq x \leq 1$.

Notice that outside the boundary layers at $x = 0$ and $x = 1$ the solution is exponentially small.

To find the term $O(\epsilon)$ in (62) one needs, respectively, the terms $y_1$ and $Y_1$ of the expansions $y_{out}(x; \epsilon)$ and $y_{in,1}(\xi; \epsilon)$ and $Y_1$ and $Y_2$ of the expansion $y_{in,0}(\xi; \epsilon)$ because the boundary layer at $x = 0$ has width $\delta = \epsilon^{1/2}$. This is left as exercise for the interested reader.

5. Nested Layers

The singular layers we have encountered so far all have a simple structure. The structure of a singular layer, however, need not be simple as shown by the following example.
Example 5.1. Consider the differential equation:

\[ \epsilon^3 x y'' + x^2 y' - (x^3 + \epsilon) y = 0, \quad 0 \leq x \leq 1, \]
\[ y(0; \epsilon) = 1, \quad y(1; \epsilon) = \sqrt{\epsilon}, \quad (63) \]

where \( \epsilon \) is a small positive parameter. The coefficient \( a(x) = x^2 \) of the first derivative \( y' \) is positive on any interval \( 0 < x_0 \leq x \leq 1 \); from Example 2.3 we know that this rules out a boundary layer at \( x = 1 \) and implies a boundary layer at \( x = x_0 \). Thus, since \( x_0 \) can be arbitrarily small, the solution of the differential equation (63) exhibits a boundary layer at \( x = 0 \). The coefficient \( a(x) \), however, vanishes at \( x = 0 \), and hence Example 2.3 does not help to determine the width of the boundary layer as \( \epsilon \to 0^+ \). We postpone the discussion of the boundary layer width after the analysis of the outer expansion.

Setting \( \epsilon = 0 \) in (63) we obtain the differential equation

\[ x^2 y_0' - x^3 y_0 = 0, \]

for the leading order term \( y_0(x) \) of the outer expansion \( y_{\text{out}}(x; \epsilon) \) valid for \( x \) not too close to the boundary \( x = 0 \). The solution to this equation with the boundary condition \( y_0(1) = y(1; \epsilon) = \sqrt{\epsilon} \) is

\[ y_0(x) = e^{x^2/2}. \]

Notice that \( y_0(0) \) is well defined; the singular nature of the expansion shows up only to order \( \epsilon \). Substituting \( y_{\text{out}}(x; \epsilon) = y_0(x) + y_1(x)\epsilon + O(\epsilon^2) \) into the differential equation (63) leads to the differential equation:

\[ x^2 y_1' - x^3 y_1 = y_0, \]

for the next-to-leading order term \( y_1(x) \). Inserting the explicit form of \( y_0(x) \), and dividing the result by \( x^2 \), gives

\[ y_1' - xy_1 = \frac{1}{x^2} e^{x^2/2}. \]

valid for \( x > 0 \). The solution to this differential equation is

\[ y_1(x) = a_1 e^{x^2/2} - \frac{1}{x} e^{x^2/2}, \]

with \( a_1 = 1 \) fixed by the condition \( y_1(1) = 0 \), so that

\[ y_{\text{out}}(x; \epsilon) \sim \left[ 1 + \epsilon \left( 1 - \frac{1}{x} \right) \right] e^{x^2/2} + O(\epsilon^2), \quad \epsilon \to 0^+. \quad (64) \]

The next-to-leading-order term \( y_1(x) \) of \( y_{\text{out}}(x; \epsilon) \) diverges as \( x \to 0^+ \), signalling the presence of the boundary layer at \( x = 0 \). The asymptotic expansion (64) is, indeed, uniform on \( \epsilon \ll x_0 \leq x \leq 1 \) because when \( x \) becomes so small that \( \epsilon/x = O(1) \) then \( \epsilon y_1 = O(y_0) \) as \( \epsilon \to 0^+ \).
To study the boundary layer at $x = 0$ we introduce the inner variable $\xi = x/\delta$ and rewrite the differential equation (63) in terms of $\xi$:

$$\frac{\epsilon^3}{\delta} \xi y'' + \delta^2 \xi y' - (\delta^3 \xi^3 + \epsilon) y = 0.$$  \hspace{1cm} (65)

Balancing the first term with the last term leads to the distinguished limit $\epsilon^3/\delta \sim \epsilon$ as $\epsilon \to 0^+$, which gives that $\delta = \epsilon^2$. A boundary layer of width $\delta = \epsilon^2$, however, poses some problems because the outer expansion $y_{\text{out}}(x; \epsilon)$ ceases to be an uniform expansion as $x = O(\epsilon)$, and hence it cannot be matched with the inner expansion $y_{\text{in}}(\xi; \epsilon)$ valid inside the boundary layer of width $\delta = \epsilon^2 = o(\epsilon)$ as $\epsilon \to 0^+$.

A closer inspection of (65) reveals a second distinguished limit, obtained by balancing the second term with the last term, consistent with $\delta \ll 1$ as $\epsilon \to 0^+$; this is given $\delta = \epsilon$.

Which of the two we take $\delta = \epsilon^2$ or $\delta = \epsilon$? If we take $\delta = \epsilon^2$ we cannot match the inner and the outer expansions; on the other hand, if we take $\delta = \epsilon$ the higher derivative $y'''$ disappears when $\epsilon = 0$ and we cannot simultaneously match $y_{\text{in}}$ with $y_{\text{out}}$ and satisfy the boundary condition at $x = 0$. However, we can match the outer expansion $y_{\text{out}}$ with the inner expansion $y_{\text{in}, \epsilon}$ valid inside the boundary layer of width $\delta = \epsilon$, then reach $x = 0$ by matching $y_{\text{in}, \epsilon}$ with the inner expansion $y_{\text{in}, \epsilon^2}$ valid inside the boundary layer of width $\delta = \epsilon^2$. In other words, if this strategy works, the boundary layer at $x = 0$ is made by a first thin layer of width $\delta = \epsilon^2$ followed by a thicker boundary of width $\delta = \epsilon$.

Consider first the case $\delta = \epsilon$. Setting $\delta = \epsilon$ in (65), and dividing the result by $\epsilon$, leads to the differential equation

$$\epsilon \xi y''' + \xi^2 y' - (\epsilon^2 \xi^3 + 1) y = 0,$$

where $\xi_1 = x/\epsilon$. The leading order term $Y_0(\xi_1)$ of the inner expansion $y_{\text{in}, \epsilon}(\xi_1; \epsilon)$ as $\epsilon \to 0^+$ is given by the solution of the differential equation,

$$\xi_1^2 Y_0' = Y_0 = 0,$$

and reads:

$$Y_0(\xi_1) = A_0 e^{-1/\xi_1}.$$  \hspace{1cm} (66)

Thus

$$y_{\text{in}, \epsilon}(\xi_1, \epsilon) \sim A_0 e^{-1/\xi_1} + O(\epsilon), \quad \epsilon \to 0^+,$$

and is valid uniformly for $\epsilon \ll \xi_1 \ll 1/\epsilon$. The constant $A_0$ is determined by the matching of $y_{\text{in}, \epsilon}(\xi_1; \epsilon)$ for $\xi_1 \gg 1$ with $y_{\text{out}}(x; \epsilon)$ for $x \ll 1$. Using (64) the inner limit of $y_{\text{out}}(x; \epsilon)$ to $O(1)$ is:

$$y_{\text{out}}(x; \epsilon) \bigg|_{\text{in}} \sim 1 + O(x^2) + O(\epsilon), \quad x \ll 1,$$

and is valid in the domain $I_1 : \epsilon^{1/2} \ll x \ll 1$ as $\epsilon \to 0^+$. 32
On the other hand, from (66) we find that the outer limit of $y_{\text{in},\epsilon}(\xi_1;\epsilon)$ to $O(1)$ as $\xi_1 \gg 1$ is:

$$y_{\text{in},\epsilon}(\xi_1;\epsilon)|_{\text{out}} \sim A_0 + O(1/\xi_1) + O(\epsilon), \quad \xi_1 \gg 1 \quad (67)$$

and is valid in the domain $I_\epsilon : 1 \ll \xi_1 \ll 1/\epsilon$.

Asymptotic matching to $O(1)$ of $y_{\text{out}}(x;\epsilon)$ and $y_{\text{in},\epsilon}(\xi_1;\epsilon)$ in the intermediate region $D : I_1 \cap I_\epsilon = \epsilon^{1/2} \ll x \ll 1$ requires that the $\delta$-limit

$$\lim_{\epsilon \to 0} [y_{\text{out}}(x;\epsilon) - y_{\text{in},\epsilon}(x/\epsilon;\epsilon)] = 1 - A_0,$$

with $x = \zeta_\delta\delta$, where $\zeta_\delta$ is fixed and $\epsilon^{1/2} \ll \delta \ll 1$, must vanish. Thus $A_0 = 1$ and

$$y_{\text{in},\epsilon}(\xi_1;\epsilon) \sim e^{-1/\xi_1} + O(\epsilon), \quad \epsilon \to 0^+. \quad (68)$$

Note that $y_{\text{in},\epsilon}(\xi_1;\epsilon)$ cannot satisfy the boundary value $y(0,\epsilon) = 1$ because it vanishes exponentially as $\xi_1 \to 0^+$.

We now examine the boundary layer of width $\delta = \epsilon^2$. Setting $\delta = \epsilon^2$ in (65), and dividing the result by $\epsilon$, we arrive at the differential equation,

$$\xi_2 y'' + \epsilon \xi_2 y' - \left(\epsilon^5 \xi_2^3 + 1\right) y = 0 \quad (69)$$

where $\xi_2 = x/\epsilon^2$. Thus, the leading order term $Y_0(\xi_2)$ of the inner expansion $y_{\text{in},\epsilon}(\xi_2;\epsilon)$ valid inside the boundary layer width $\delta = \epsilon^2$ satisfies the differential equation,

$$\xi_2 Y_0'' - Y_0 = 0, \quad (69)$$

with the boundary condition $Y_0(0) = y(0;\epsilon) = 1$. This differential equation cannot be solved in terms of elementary functions; its solution, however, can be expressed in terms of the modified Bessel functions $K_\nu$ and $I_\nu$. Writing $Y_0(\xi_2)$ as

$$Y_0(\xi_2) = \frac{s}{2} f(s),$$

with $s = 2\sqrt{\xi_2}$, and using

$$\frac{d}{d\xi_2} Y_0(\xi_2) = \frac{2}{s} \frac{d}{ds} \left[ \frac{s}{2} f(s) \right] = \frac{1}{s} f(s) + f'(s),$$

$$\frac{d^2}{d\xi_2^2} Y_0(\xi_2) = \frac{2}{s} \frac{d}{ds} \left[ \frac{1}{s} f(s) + f'(s) \right] = -\frac{2}{s^3} f(s) + \frac{2}{s^2} f'(s) + \frac{2}{s} f''(s),$$

the differential equation (69) becomes

$$s^2 f'' + sf' - (s^2 + 1)f = 0. \quad (70)$$

The solution to this differential equation is a linear combination of the modified Bessel functions $K_\nu(s)$ and $I_\nu(s)$ of index $\nu = 1$ solutions of

$$s^2 f'' + sf' - (s^2 + \nu^2)f = 0.$$
Thus
\[ Y_0(\xi_2) = A_0 \sqrt{\xi_2} K_1(2 \sqrt{\xi_2}) + B_0 \sqrt{\xi_2} I_1(2 \sqrt{\xi_2}), \]
where \( A_0 \) and \( B_0 \) are determined from the boundary condition at \( x = 0 \) and the asymptotic matching with \( y_{in,\epsilon}(\xi_1; \epsilon) \) as \( \xi_2 \to 1 \). When \( \xi_2 \gg 1 \) the modified Bessel function \( I_1(2 \sqrt{\xi_2}) \sim e^{2 \sqrt{\xi_2}} \) while \( K_1(2 \sqrt{\xi_2}) \sim e^{-2 \sqrt{\xi_2}} \); matching hence requires \( B_0 = 0 \) and \( y_{in,\epsilon}(\xi_2; \epsilon) \) is exponentially small in the domain \( I, 2 : 1 \ll 1/\epsilon^2 \) as \( \epsilon \to 0^+ \). On the other hand, the inner solution \( y_{in,\epsilon}(\xi_1; \epsilon) \) is exponentially small in the domain \( I : \epsilon \ll 1 \ll 1 \). Thus, since \( y_{in,\epsilon}(\xi_2; \epsilon) \) and \( y_{in,\epsilon}(\xi_1; \epsilon) \) are both \( o(1) \) as \( \epsilon \to 0^+ \) in the intermediate domain \( D : I_2 \cap I_\epsilon = \epsilon^2 \ll 1 \), they match asymptotically to \( O(1) \) in \( D \). The width of the intermediate region vanishes as \( \epsilon \to 0^+ \), nevertheless \( \delta_2/\delta_1 = \epsilon/\epsilon^2 \gg 1 \) as \( \epsilon \to 0^+ \) and the asymptotic matching is well defined.

The constant \( A_0 \) is determined by the boundary condition \( Y_0(\xi_2) = 0 \). To evaluate the value of \( Y_0(\xi_2) = 0 \) we need the behavior of \( K_1(2 \sqrt{\xi_2}) \) for \( \xi_2 \ll 1 \). This can be obtained directly from (70) because when \( s \ll 1 \) the term \( s^2 \) can be neglected compared to 1 and the differential equation is solved by \( f = 1/s \). Thus \( K_1(s) \sim 1/s + o(1/s) \) as \( s \to 0 \) and
\[ \lim_{\xi_2 \to 0} Y_0(\xi_2) = A_0 \lim_{\xi_2 \to 0} \sqrt{\xi_2} K_1(2 \sqrt{\xi_2}) = A_0/2. \]
The request \( Y_0(\xi_2) = 0 \) fixes \( A_0 = 2 \) and \( y_{in,\epsilon}(\xi_2; \epsilon) \) reads:
\[ y_{in,\epsilon}(\xi_2; \epsilon) \sim 2 \sqrt{\xi_2} K_1(2 \sqrt{\xi_2}) + O(\epsilon), \quad \epsilon \to 0^+. \]
This expansion is uniform on \( 0 \leq \xi_2 \ll 1/\epsilon^2 \).

Finally, combining the expansions \( y_{out}(x; \epsilon), y_{in,\epsilon}(\xi_1; \epsilon) \) and \( y_{in,\epsilon}(\xi_2; \epsilon) \) with \( y_{match,\epsilon}(x; \epsilon) = 1 \) and \( y_{match,\epsilon}(x; \epsilon) = 0 \):
\[ y_{unif}(x; \epsilon) \sim y_{out}(x; \epsilon) + y_{in,\epsilon}(x/\epsilon; \epsilon) - y_{match,\epsilon}(x; \epsilon) + y_{in,\epsilon}(x/\epsilon^2; \epsilon) - y_{match,\epsilon}(x; \epsilon), \]
we obtain the uniform asymptotic expansion
\[ y_{unif}(x; \epsilon) \sim e^{x^2/2} + e^{-\epsilon x} - 1 + 2 \sqrt{\xi} K_1(2 \sqrt{\xi}/\epsilon) + O(\epsilon), \quad \epsilon \to 0^+, \]
which gives an approximation differing from the exact solution of the differential equation (63) by term \( o(1) \) as \( \epsilon \to 0^+ \) uniformly on the whole interval \( 0 \leq x \leq 1 \).

6. Internal Layers

Singular layers may occur at the edge(s) of the domain of interest, as in all the examples discussed, or at its interior. In this case one speaks of internal layer. The study of problems with internal layers is usually more complex. An internal singular layer can be moved to the boundary by partitioning the domain of interest into sub-domains; thus an internal layer problem can be treated by
studying the boundary layer problem in each sub-domain and then combine the single expansions to form a unique uniform expansion valid in the original domain.

**Example 6.1.** Consider the differential equation:

\[\epsilon y'' + 2xy' + (1 + x^2)y = 0, \quad -1 \leq x \leq 1,\]
\[y(-1; \epsilon) = 2, \quad y(1; \epsilon) = 1,\]

where \(\epsilon\) is a small positive parameter. This differential equation is of the type discussed in Example 2.3 with \(a(x) = 2x\) and \(b(x) = 1 + x^2\). Both \(a(x)\) and \(b(x)\) are bounded in \(-1 \leq x \leq 1\), however \(a(x)\) vanishes at \(x = 0\) and hence the results of Example 2.3 cannot be applied straightforwardly. Nevertheless, from Example 2.3 we know that if we restrict to the domain \(D_R(x_0): 0 < x_0 \leq x \leq 1\) there is a boundary layer at \(x = x_0\). Similarly, since \(a(x) < 0\) for \(-1 \leq x \leq -x_1 < 0\), in the domain \(D_L(x_1): -1 \leq x \leq -x_1 < 0\) there is a boundary layer at \(x = -x_1\). Both \(x_0\) and \(x_1\) can be arbitrarily small; thus, since \(D_L(0^-) \cup D_R(0^+) = -1 \leq x \leq 1\), the solution of the differential equation (71) has an internal layer at \(x = 0\).

An uniform approximation as \(\epsilon \to 0^+\) can be constructed by studying separately the problem in the left domain \(D_L(0^-)\) and in the right domain \(D_R(0^+)\) and then combining the results into a single uniform expansion.

We begin the analysis by studying the leading order term \(y_0(x; \epsilon)\) of the outer expansion \(y_{out}(x; \epsilon)\) as \(\epsilon \to 0^+\). Setting \(\epsilon = 0\) into the differential equation (71) we obtain

\[2xy'' + (1 + x^2)y = 0,\]

whose solution is

\[y_0(x) = \frac{a_0}{\sqrt{|x|}} e^{-x^2/4}.\]

The leading order term of \(y_{out}(x; \epsilon)\) diverges as \(x \to 0\), signaling the presence of a singular layer at \(x = 0\). The value of the constant \(a_0\) is determined by the boundary condition. The condition \(y_0(-1) = 2\) gives \(a_0 = 2\epsilon^2\), while imposing \(y_0(1) = 1\) leads to \(a_0 = e\). Thus, the outer expansion to \(O(1)\) is

\[y_{out,L}(x; \epsilon) \sim \frac{2}{\sqrt{-x}} e^{(1-x^2)/4} + O(\epsilon), \quad \epsilon \to 0^+,\]

for \(x < 0\) and

\[y_{out,R}(x; \epsilon) \sim \frac{1}{\sqrt{x}} e^{(1-x^2)/4} + O(\epsilon), \quad \epsilon \to 0^+,\]

for \(x > 0\). Both expansions are uniform for \(x\) not too close to \(x = 0\), and hence they cannot be matched directly. Matching can only occur through the singular layer at \(x = 0\).

The width \(\delta\) of the singular layer is not \(O(\epsilon)\) as \(\epsilon \to 0^+\) because \(a(x)\) vanishes at \(x = 0\). Rewriting the differential equation (71) in terms of the inner variable \(\xi = x/\delta\) we obtain:

\[\frac{\epsilon}{\delta} \xi'' + 2\xi y' + (1 + \epsilon\xi^2)y = 0.\]
Balancing the first term with the second and third, the last term is \( O(\epsilon) \) as \( \epsilon \to 0^+ \), gives the distinguished limit \( \epsilon/\delta^2 \sim 1 \) as \( \epsilon \to 0^+ \). Thus, the layer has width \( \delta = \epsilon^{1/2} \) and the differential equation in the inner variable then becomes

\[
y'' + 2\xi y' + (1 + \epsilon \xi^2) y = 0.
\]

The leading order term \( Y_0(\xi) \) of the inner expansion \( y_{in}(\xi;\epsilon) \) as \( \epsilon \to 0^+ \) is obtained by solving the differential equation:

\[
Y_0'' + 2\xi Y_0' + Y_0 = 0, \tag{74}
\]

and can be expressed in term of the \textit{Parabolic Cylindric Functions} \( D_\nu(s) \) solution of the differential equation

\[
f'' + \left( \nu + \frac{1}{2} - \frac{s^2}{4} \right) f = 0. \tag{75}
\]

Indeed, setting

\[
Y_0(\xi) = e^{-s^2/4} f(s).
\]

with \( s = \sqrt{2} \xi \), and using

\[
\frac{d}{d\xi} Y_0 = \sqrt{2} \left( f' - \frac{s}{2} f \right) e^{-s^2/4},
\]

\[
\frac{d^2}{d\xi^2} Y_0 = 2 \left[ f'' - sf' + \frac{1}{2} \left( \frac{s^2}{2} - 1 \right) f \right] e^{-s^2/4},
\]

the differential equation (74) becomes:

\[
f'' - \frac{s^2}{4} f = 0,
\]

which is the particular case of (75) with \( \nu = -1/2 \). The differential equation (75) remains unchanged if \( s \to -s \), thus \( Y_0(\xi) \) is given by a linear combination of \( D_{-1/2}(s) \) and \( D_{-1/2}(-s) \):

\[
Y_0(\xi) = e^{-s^2/4} \left[ A_0 D_{-1/2}(\sqrt{2} \xi) + B_0 D_{-1/2}(-\sqrt{2} \xi) \right].
\]

The constants \( A_0 \) and \( B_0 \) are determined by matching the inner expansion \( y_{in}(\xi;\epsilon) = Y_0(\xi) + O(\epsilon) \) with the outer expansions \( y_{out, L}(x;\epsilon) \) and \( y_{out, R}(x;\epsilon) \).

The inner limit to \( O(1) \) of \( y_{out, R}(x;\epsilon) \) is:

\[
y_{out, R}(x;\epsilon) \bigg|_{\text{in}} \sim e^{1/4} x^{-1/2} + O(x^{3/2}) + O(\epsilon), \quad x \ll 1,
\]

and is valid in the domain \( I_1 : \epsilon^{2/3} \ll x \ll 1 \) as \( \epsilon \to 0^+ \).

To evaluate the outer limit of \( y_{in}(\xi;\epsilon) \) we use the asymptotic behavior:

\[
D_\nu(s) \sim s^\nu e^{-s^2/4}, \quad s \to +\infty,
\]

\[
D_\nu(-s) \sim \frac{\sqrt{2\pi}}{\Gamma(-\nu)} s^{-\nu-1} e^{-s^2/4}, \quad s \to +\infty,
\]

\[36\]
of the parabolic cylinder functions $\text{D}_p(s)$ for large positive and negative argument. The right outer limit of $y_{\text{in}}(\xi; \epsilon)$ to $O(1)$ then reads:

$$
y_{\text{in}}(\xi; \epsilon) \bigg|_{\text{out}} \sim B_0 \left( \sqrt{2} \xi \right)^{-\frac{1}{2}} \frac{\sqrt{2\pi}}{\Gamma(1/2)} + O(\epsilon^2) + O(\epsilon),
$$

and is valid in the domain $I_0^+ : 1 \ll \xi \ll 1/\epsilon^{1/2}$.

Consequently the intermediate limit to $O(1)$ of $y_{\text{out}, R}(x; \epsilon)$ and $y_{\text{in}}(\xi; \epsilon)$ in the domain $D = I_1 \cap I_0^+ : \epsilon^{1/2} \ll x \ll 1$ gives:

$$
\lim_{\epsilon \to 0^+} \left[ y_{\text{out}, R}(x; \epsilon) - y_{\text{in}}(x/\epsilon^{1/2}; \epsilon) \right] = [\epsilon^{1/4} - B_0(2\epsilon)^{1/4}] x^{-1/2},
$$

and fixes $B_0 = (e/2\epsilon)^{1/4}$. The $\delta$-limit is taken setting $x = \zeta_\delta \delta$ with $\zeta_\delta$ fixed and $\epsilon^{1/2} \ll \delta \ll 1$ as $\epsilon \to 0^+$.

The inner limit of $y_{\text{out}}(x; \epsilon)$ to $O(1)$ is:

$$
y_{\text{out}, L}(x; \epsilon) \bigg|_{\text{in}} \sim 2 \epsilon^{1/4} (-x)^{-1/2} + O(|x|^{3/2}) + O(\epsilon), \quad -x \ll 1,
$$

and is valid in the domain $I_{-1} : e^{2/3} \ll -x \ll 1$ as $\epsilon \to 0^+$.

On the other hand, the left outer limit of $y_{\text{in}}(\xi; \epsilon)$ to $O(1)$ reads:

$$
y_{\text{in}}(\xi; \epsilon) \bigg|_{\text{out}} \sim A_0 \left( -\sqrt{2} \zeta \right)^{-\frac{1}{2}} \frac{\sqrt{2\pi}}{\Gamma(1/2)} + O(\epsilon^2) + O(\epsilon), \quad -\xi \gg 1,
$$

and its domain of validity is $I_0^- : 1 \ll -\xi \ll 1/\epsilon^{1/2}$.

Thus, taking the intermediate limit to $O(1)$ of $y_{\text{out}, L}(x; \epsilon)$ and $y_{\text{in}}(\xi; \epsilon)$ in the domain $D = I_{-1} \cap I_0^- : \epsilon^{1/2} \ll -x \ll 1$ we get

$$
\lim_{\epsilon \to 0^+} \left[ y_{\text{out}, L}(x; \epsilon) - y_{\text{in}}(x/\epsilon^{1/2}; \epsilon) \right] = [2\epsilon^{1/4} - A_1(2\epsilon)^{1/4}] (-x)^{-1/2},
$$

so that $A_0 = 2(e/2\epsilon)^{1/4}$.

In conclusion, the inner expansion reads:

$$
y_{\text{in}}(\xi; \epsilon) \sim \left( \frac{\epsilon}{2\epsilon} \right)^{\frac{1}{4}} \left[ 2D_{-1/2}(\sqrt{2} \xi) + D_{-1/2}(-\sqrt{2} \xi) \right] e^{-\xi^2/2} + O(\epsilon), \quad \epsilon \to 0^+. \quad (76)
$$

Combining the inner expansion with the left, or right, outer expansion and recalling that

$$
y_{\text{match, R}}(x; \epsilon) \sim \epsilon^{1/4} x^{-1/2} + O(\epsilon),
\quad y_{\text{match, L}}(x; \epsilon) \sim 2 \epsilon^{1/4} (-x)^{-1/2} + O(\epsilon), \quad \epsilon \to 0^+.
$$

we get

$$
y_{\text{unif, L}}(x; \epsilon) \sim \left( \frac{\epsilon}{2\epsilon} \right)^{\frac{1}{4}} \left[ 2D_{-1/2}(x\sqrt{2}/\epsilon) + D_{-1/2}(-x\sqrt{2}/\epsilon) \right] e^{-x^2/2\epsilon}
+ \frac{2\epsilon^{1/4}}{\sqrt{-x} \epsilon} \left[ e^{-x^2/4} - 1 \right] + O(\epsilon), \quad x \leq 0, \quad \epsilon \to 0^+. \quad (77)
$$
and
\[
y_{\text{unif}, R}(x; \epsilon) \sim \left( \frac{e}{2\epsilon} \right)^{1/4} \left[ 2D_{-1/2}(x\sqrt{2/\epsilon}) + D_{-1/2}(-x\sqrt{2/\epsilon}) \right] e^{-x^2/2\epsilon} + \frac{e^{1/4}}{\sqrt{x}} \left[ e^{-x^2/4} - 1 \right] + O(\epsilon), \quad x \geq 0, \quad \epsilon \to 0^+.
\] (78)

The expansions (77) and (78) give an asymptotic expansion of the solution of the differential equation (71) that is valid uniformly to $O(1)$ as $\epsilon \to 0^+$ on the whole interval $0 \leq x \leq 1$.

The difference between the exact solution of the differential equation (71) and the expansions (77) and (77) is $o(1)$ as $\epsilon \to 0^+$; hence, neglecting terms $o(1)$ as $\epsilon \to 0^+$, expansions (77) and (78) can be combined into the more compact form
\[
y_{\text{unif}}(x; \epsilon) \sim \left[ \frac{e^{(1-x^2)}}{2\epsilon} \right]^{1/4} \left[ 2D_{-1/2}(x\sqrt{2/\epsilon}) + D_{-1/2}(-x\sqrt{2/\epsilon}) \right] e^{-x^2/2\epsilon} + O(\epsilon).
\] (79)

We stress, however, that (77)-(78) and (79) are only asymptotically equivalent to $O(1)$ as $\epsilon \to 0^+$, for finite $\epsilon$ they give different numerical values.