# Field Theory of Neural Networks
# A Tutorial (Part 3)

- https://www2.phys.uniroma1.it/doc/crisanti/Teach/DFT/Files/CNS-19.pdf

# Outline

- Model
- Dynamic Field Theory
- Disorder Average
- Dynamic Mean Field Theory
- Stability and Replicas
- Chaotic Solution and Lyapunov Exponent


- H. Sompolinsky, A. C., H.-J. Sommers
  Phys. Rev. Lett. 61, 259 (1988)
- A. C, H. Sompolinsky,
  Phys. Rev. E. 98, 062120 (2018)

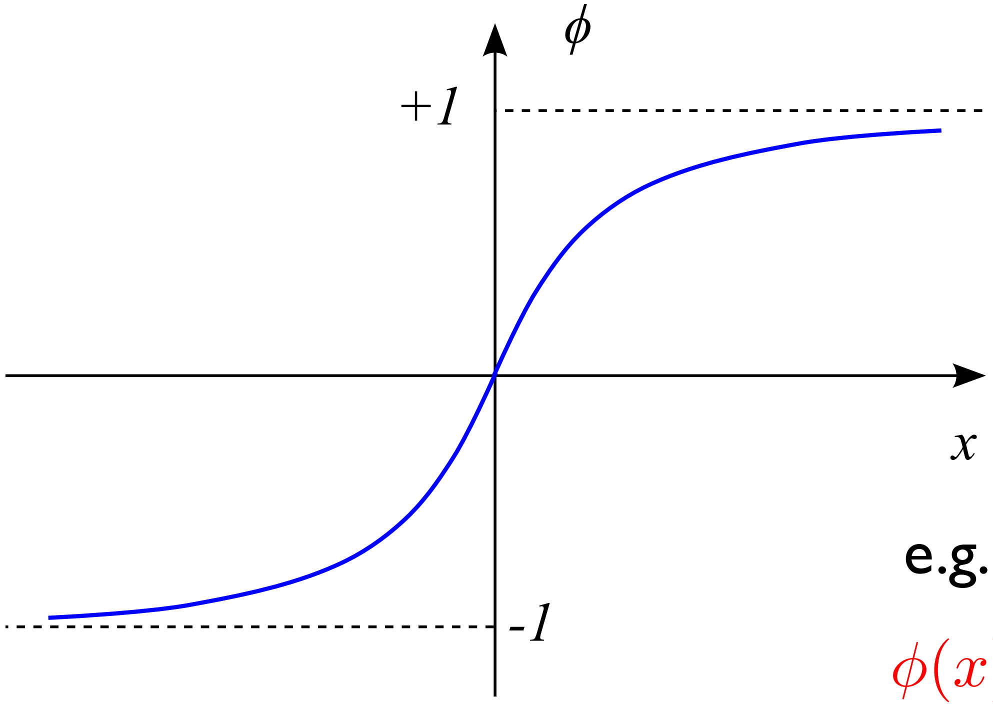
# The Model

- Spin or "Neuron"

$$S_i(t) = \phi\big(gh_i(t)\big)$$

$g > 0$ gain parameter

$$\phi(-x) = -\phi(x)$$
$$\phi(\pm\infty) = \pm 1$$




e.g.

$$\phi(x) = \tanh(x)$$

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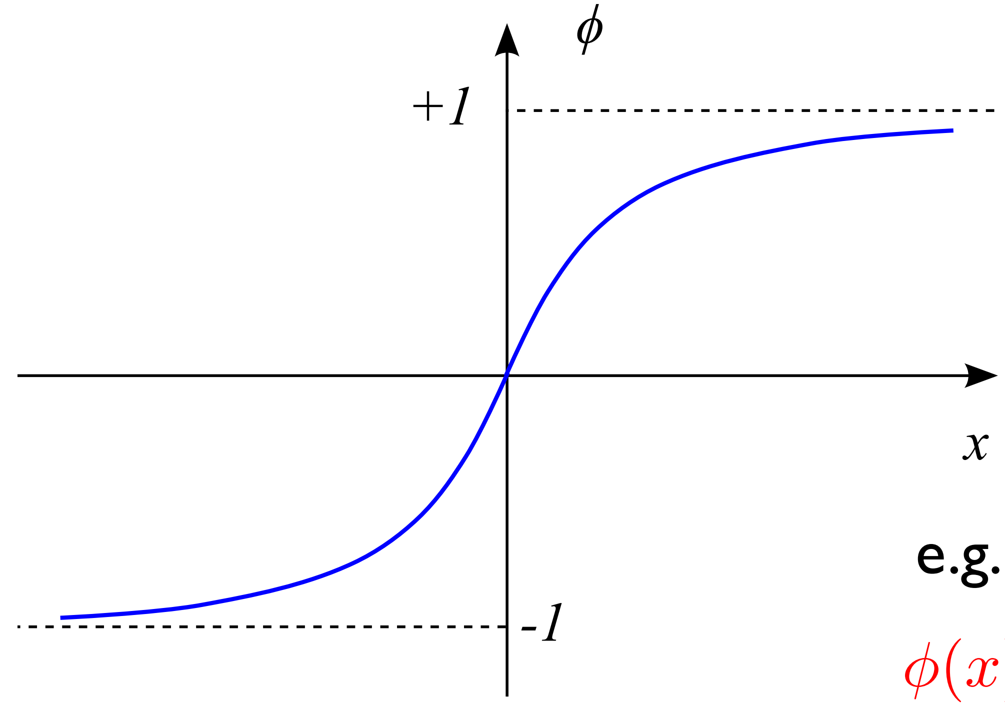


e.g.

$$\phi(x) = \tanh(x)$$

- Dynamics

$$\frac{d}{dt}h_i(t) = -h_i(t) + \sum_{j=1}^{N} J_{ij} S_j(t) + \rho_i(t)$$

$$h_i(t_0) = h_i^0$$

Kirchhoff's Equation

External source (field)

$J_{ij} :=$ "Synaptic" matrix

$$\left[J_{ii} = 0\right]$$

# The Model

Our goal is to describe the dynamical behavior of the network

- Dynamics depends on the Synaptic Matrix

- *symmetric* $J_{ij}$ → Relaxation of an Energy Function $E(h_1, \ldots, h_n)$

  Dynamics is "*simple*"

  *Stable fix points local minima of Energy*

- *non-symmetric* $J_{ij}$ → No Energy Function

  Dynamics is *non-trivial*

  *Fix points, Limit Cycles, Chaotic behaviour*

# The Model

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- *non-symmetric* $J_{ij}$ → No Energy Function

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- **Assumptions**

| | |
|---|---|
| $t_0 \to -\infty$ | Steady state |
| $N \gg 1$ | Large System (Mean Field, Loop Expansion,…) |
| $J_{ij}$ | Random Matrix |

# Dynamic Field Theory in a Nutshell

Consider:

$$\longrightarrow \quad \partial_a h_i^a = F_i[h^a] + h_i^0\,\delta(t_a - t_0)$$

Initial Condition

$$F_i[h^a] = -h_i^a + \sum_{j=1}^{N} J_{ij}\,S(h_j^a) + \rho_i(t)$$

Useful notation

$$h_i^a = h_i(t_a)$$

$$\partial_a = \frac{d}{dt_a} + \delta \quad (\delta \to 0^+)$$

Causality

Correlation functions: $\langle h_i^a\, h_j^b \cdots \rangle$

Average over initial condition, disorder, etc…

Response functions: $\frac{\delta}{\delta \rho_k^c}\frac{\delta}{\delta \rho_l^d}\cdots\ \langle h_i^a\, h_j^b \cdots \rangle_\rho$ (Susceptibilities)

# Dynamic Field Theory in a Nutshell

Consider:

$$\longrightarrow \quad \partial_a h_i^a = F_i[h^a] + h_i^0\, \delta(t_a - t_0)$$

$$F_i[h^a] = -\, h_i^a + \sum_{j=1}^{N} J_{ij}\, S(h_j^a) + \rho_i(t)$$

Useful notation

$$h_i^a = h_i(t_a)$$

$$\partial_a = \frac{d}{dt_a} + \delta \quad (\delta \to 0^+)$$

Path Integral Approach $\rightarrow$

Generating Functional for Correlation and Response Functions induced by the dynamics

# Dynamic Field Theory in a Nutshell

a) Discretize time: dynamical equation → finite difference equation

$$\partial_a h_i^a = F_i[h^a] + h_i^0\,\delta(t_a - t_0) \longrightarrow h_i^{a+1} - h_i^a = F_i[h^a]\,\delta t + h_i^0\,\delta_{a,0}^{\mathrm{Kr}}$$

Kronecker delta

b) Generating Functional: sum $e^{i\hat{b}_i^a h_i^a}$ over all path (trajectories)

$$Z[\hat{b}] = \int \prod_{i,a} dh_i^a\,\delta\left(h_i^a - \tilde{h}_i^a\right)\,e^{i\hat{b}_i^a h_i^a}$$

Solution of the finite difference equation

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Solution of the finite difference equation

- Derivatives $i\hat{b}$ → Products of $h$ → $\hat{b} = 0$ → Correlation Functions

Note: $Z[\hat{b} = 0] = 1$

# Dynamic Field Theory in a Nutshell

a) Discretize time: dynamical equation → finite difference equation

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Solution of the finite difference equation

$$h_i^a - \tilde{h}_i^a = 0 \quad \longrightarrow \quad f(h_i^a) = h_i^{a+1} - h_i^a - F_i[h^a]\,\delta t - h_i^0\,\delta^{\mathrm{Kr}}_{a,0} = 0$$

$$\longrightarrow \quad \delta\left(h_i^a - \tilde{h}_i^a\right) = |f'(h_i^a)|\,\delta(f(h_i^a))$$

Ito scheme → Jacobian $= 1$

Jacobian transformation: $h_i^a - \tilde{h}_i^a = 0 \rightarrow f(h_i^a) = 0$

# Dynamic Field Theory in a Nutshell

a) Discretize time: dynamical equation → finite difference equation

$$\partial_a h_i^a = F_i[h^a] + h_i^0\,\delta(t_a - t_0) \quad \longrightarrow \quad h_i^{a+1} - h_i^a = F_i[h^a]\,\delta t + h_i^0\,\delta_{a,0}^{\mathrm{Kr}}$$

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Solution of the finite difference equation

$$= \int \prod_{i,a} dh_i^a\,\delta\left(h_i^{a+1} - h_i^a - F_i[h^a]\delta t - h_i^0\delta_{a,0}^{\mathrm{Kr}}\right)\,e^{i\hat{b}_i^a h_i^a}$$

# Dynamic Field Theory in a Nutshell

c) Integral Representation of $\delta$ Function: hat-fields

$$\delta(z_i^a) = \int_{-\infty}^{+\infty} \frac{d\hat{h}_i^a}{2\pi} e^{-i\hat{h}_i^a z_i^a}$$

$$\longrightarrow \quad Z[\hat{b}] = \int \prod_{i,a} \frac{d\hat{h}_i^a \, dh_i^a}{2\pi} \exp\left[-i\hat{h}_i^a \left(h_i^{a+1} - h_i^a - F_i[h_a]\delta t - h_i^0 \delta_{a,0}^{\mathrm{Kr}}\right) + i\hat{b}_i^a h_i^a\right]$$

# Dynamic Field Theory in a Nutshell

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$$\delta(z_i^a) = \int_{-\infty}^{+\infty} \frac{d\hat{h}_i^a}{2\pi} e^{-i\hat{h}_i^a z_i^a}$$

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d) Add source for hat-field and take continuum limit $\delta t \to 0$

$$Z[\hat{b}, b] = \int \mathscr{D}\hat{h}_i \, \mathscr{D}h_i \, e^{-S[\hat{h},h] + \sum_{ia}\left(i\hat{b}_i^a h_i^a + i\hat{h}_i^a b_i^a\right)}$$

$$S[\hat{h}, h] = \sum_{i,a} i\hat{h}_i^a \left[\partial_a h_i^a - F_i[h^a] - h_i^0 \delta(t_a - t_0)\right]$$

$$\mathscr{D}h_i = \lim_{\delta t \to 0} \prod_a dh_i^a$$

$$\mathscr{D}\hat{h}_i = \lim_{\delta t \to 0} \prod_a \frac{d\hat{h}_i^a}{2\pi}$$

$$\sum_a = \int_{t_0}^{t} dt_a$$

# Dynamic Field Theory in a Nutshell

- hat-fields $\hat{h}$ $\rightarrow$ Response fields

$$Z[\hat{b}, b] = \int \mathscr{D}\hat{h}_i \, \mathscr{D}h_i \, e^{-S[\hat{h},h] + \sum_{ia}\left(i\hat{b}_i^a h_i^a + i\hat{h}_i^a b_i^a\right)}$$

$$S[\hat{h}, h] = \sum_{i,a} i\hat{h}_i^a \left[\partial_a h_i^a - F_i[h^a] - h_i^0 \delta(t_a - t_0)\right]$$

Note: $b_i^a \leftrightarrow \rho_i^a$

Derivatives $\rho_i^a$ → Derivatives $b_i^a$ → Products of $i\hat{h}_i^a$

→

$$\langle h_{i_1}^{a_1} \ldots h_{i_n}^{a_n} \hat{h}_{j_1}^{b_1} \ldots \hat{h}_{j_m}^{b_m} \rangle = \frac{\delta}{\delta b_{j_1}^{b_1}} \ldots \frac{\delta}{\delta b_{j_m}^{b_m}} \langle h_{i_1}^{a_1} \ldots h_{i_n}^{a_n} \rangle$$

Response Functions

Note: $Z[\hat{b} = 0, b] = 1$

# Dynamic Field Theory in a Nutshell

- Disorder Average

$$F_i[h^a] = -h_i^a + \sum_{j=1}^{N} J_{ij} S(h_j^a)$$

Random

→ Correlation/Response functions depend on $J_{ij}$ → Random quantities

Note: $Z[\hat{b} = 0, b] = 1$

Average Correlation/Response →

$$\overline{Z}[\hat{b}, b] = \int d\mathbf{J}\, P[\mathbf{J}]\, Z[\hat{b}, b]$$

# Dynamic Field Theory in a Nutshell

- General Matrix:

$$J_{ij} = J_{ij}^{S} + k\, J_{ij}^{A}$$

$$J_{ji}^{S} = J_{ij}^{S}$$ Symmetric part

$$J_{ji}^{A} = -\, J_{ij}^{A}$$ Antisymmetric part

# Dynamic Field Theory in a Nutshell

- General Matrix:

$$J_{ij} = J^S_{ij} + k\, J^A_{ij}$$

$J^S_{ji} = J^S_{ij}$ Symmetric part

$J^A_{ji} = -\, J^A_{ij}$ Antisymmetric part

- i.i.d Gaussian : $J_{ij}$

$$\overline{J^S_{ij}} = \overline{J^A_{ij}} = 0 \qquad \overline{(J^S_{ij})^2} = \overline{(J^A_{ij})^2} = \frac{1}{N}\frac{1}{1+k^2}$$

$\longrightarrow$

$$\overline{J_{ij}} = 0$$

$$\overline{(J_{ij})^2} = \frac{1}{N}$$

$$\overline{J_{ij}\, J_{ji}} = \frac{\gamma}{N} \qquad \gamma = \frac{1-k^2}{1+k^2}$$

$\gamma = 1$ : Symmetric Matrix

$\gamma = 0$ : Asymmetric Matrix

$\gamma = -\,1$ : Antisymmetric Matrix

$k = 0,\ 1,\ \infty$

# Dynamic Field Theory in a Nutshell

- Disorder Average

$$\overline{Z}[\hat{b}, b] \quad \Rightarrow \quad \overline{\exp\left[\sum_{aij} i\hat{h}_i^a J_{ij} S_j^a\right]}$$

$$\int \frac{dx}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma_2} \pm \mu x} = e^{\frac{\sigma^2\mu^2}{2}}$$

Just in case…

# Dynamic Field Theory in a Nutshell

- Disorder Average

$$\bar{Z}[\hat{b}, b] \quad \Rightarrow \quad \overline{\exp\left[\sum_{aij} i\hat{h}_i^a J_{ij} S_i^a\right]}$$

$$\bar{Z}[\hat{b}, b] = \int \mathscr{D}h\, \mathscr{D}\hat{h}\, \exp\Bigg\{ -\sum_{ia} i\hat{h}_i^a\,(1+\partial_a)\,h_i^a$$
$$+\frac{1}{2N}\sum_{ij}\left[\sum_a i\hat{h}_i^a S_j^a\right]^2 + \frac{\gamma}{2N}\sum_{ij}\left[\sum_a i\hat{h}_i^a S_j^a\right]\left[\sum_a i\hat{h}_j^a S_i^a\right]$$
$$+\sum_{ia}[i\hat{h}_i^a b_i^a + i\hat{b}_i^a h_i^a]\Bigg\}$$

*Averaged Correlation/Response G.F.*

# Dynamic Field Theory in a Nutshell

- Disorder Average

$$\bar{Z}[\hat{b}, b] \quad \Rightarrow \quad \overline{\exp\left[\sum_{aij} i\hat{h}_i^a J_{ij} S_i^a\right]}$$

Deterministic part

$$\bar{Z}[\hat{b}, b] = \int \mathscr{D}h\, \mathscr{D}\hat{h}\, \exp\left\{ -\sum_{ia} i\hat{h}_i^a (1 + \partial_a) h_i^a \right.$$
$$+ \frac{1}{2N} \sum_{ij} \left[\sum_a i\hat{h}_i^a S_j^a\right]^2 + \frac{\gamma}{2N} \sum_{ij} \left[\sum_a i\hat{h}_i^a S_j^a\right] \left[\sum_a i\hat{h}_j^a S_i^a\right]$$
$$\left. + \sum_{ia} [i\hat{h}_i^a b_i^a + i\hat{b}_i^a h_i^a] \right\}$$

*Averaged Correlation/Response G.F.*

Disorder induced interactions → non-local in space and time

# Dynamic Field Theory in a Nutshell

- Space Diagonalization

Define: $C^{ab} = \frac{1}{N}\sum_i S_i^a S_i^b \qquad G^{ab} = \frac{1}{N}\sum_i S_i^a i\hat{h}_i^b$

Use:
$$1 = \int \mathcal{D}X^{ab}\,\delta\left(X^{ab} - \frac{1}{N}\sum_i x^a y^b\right)$$
$$= \int \mathcal{D}X^{ab}\,\mathcal{D}(N\hat{X}^{ab})\,\exp\left[-i\hat{X}^{ab}\left(NX^{ab} - \sum_i x^a y^b\right)\right]$$

# Dynamic Field Theory in a Nutshell

- Space Diagonalization

$$\overline{Z}[\hat{b}, b] = \int \mathscr{D}\hat{C}\mathscr{D}C\mathscr{D}\hat{G}\mathscr{D}G \, e^{-N\mathcal{S}[\hat{C},C,\hat{G},G;\hat{b},b]}$$

*[Exact DFT]*

$$\mathcal{S}[\hat{C}, C, \hat{G}, G; \hat{b}, b] = \frac{1}{2}\sum_{ab}\left[i\hat{C}^{ab}C^{ab} + i\hat{G}^{ab}G^{ab}\right] - W[\hat{C}, C, \hat{G}, G; \hat{b}, b]$$

$$W[(\,\cdot\,)] = \ln \int \mathscr{D}\hat{h}\mathscr{D}h \, e^{L[\hat{h},h;\hat{C},C,\hat{G},G,\hat{b},b]}$$

$\longrightarrow$

$$L[(\,\cdot\,)] = -\sum_a i\hat{h}^a(1 + \partial_a)\, h^a + \frac{1}{2}\sum_{ab}\left[i\hat{C}^{ab}\, S^a S^b + C^{ab} i\hat{h}^a i\hat{h}^b\right] + \frac{1}{2}\sum_{ab}\left[i\hat{G}^{ab}\, S^a i\hat{h}^b + \gamma G^{ab} i\hat{h}^a S^b\right] + \sum_a \left[i\hat{h}^a b^a + i\hat{b}^a h^a\right]$$

*One Variable dynamics*

# Dynamic Field Theory in a Nutshell

- Consider

$$\sum_i \overline{\langle S_i^a S_i^b \rangle} = \int \prod_i \mathscr{D}\hat{h}_i \mathscr{D}h_i \, \overline{e^{-S[\hat{h},h]}} \sum_i S_i^a S_i^b$$

$$= \int \mathscr{D}\hat{C}\mathscr{D}C\mathscr{D}\hat{G}\mathscr{D}G \, e^{-N\mathcal{S}[\hat{C},C,\hat{G},G;0,0]} \sum_i S_i^a S_i^b$$

# Dynamic Field Theory in a Nutshell

- Consider

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$$= \int \mathscr{D}\hat{C}\mathscr{D}C\mathscr{D}\hat{G}\mathscr{D}G \, e^{-N\mathcal{S}[\hat{C},C,\hat{G},G;0,0]} \sum_i S_i^a S_i^b$$

But….

$$e^{-N\mathcal{S}[(\cdot)]} \left[ \sum_i S_i^a S_i^b \right] \to e^{-\frac{N}{2}\sum_{ab} i\hat{C}^{ab}C^{ab}+\ldots} \int \prod_i \mathscr{D}\hat{h}_i \mathscr{D}h_i \, e^{\frac{1}{2}\sum_{ab} i\hat{C}^{ab}\sum_i S_i^a S_i^b+\ldots} \left[ \sum_i S_i^a S_i^b \right]$$

$$\to e^{-\frac{N}{2}\sum_{ab} i\hat{C}^{ab}C^{ab}+\ldots} \left[ \frac{\delta}{\delta i\hat{C}^{ab}} \int \prod_i \mathscr{D}\hat{h}_i \mathscr{D}h_i \, e^{\frac{1}{2}\sum_{ab} i\hat{C}^{ab}\sum_i S_i^a S_i^b+\ldots} \right]$$

$$\to \left[ \frac{\delta}{\delta i\hat{C}^{ab}} + NC^{ab} \right] e^{-N\mathcal{S}[(\cdot)]}$$

$$\frac{\delta}{\delta i\hat{C}^{ab}} e^{-\frac{N}{2}\sum_{ab} i\hat{C}^{ab}C^{ab}} = -NC^{ab}$$

# Dynamic Field Theory in a Nutshell

- Consider

$$\sum_i \overline{\langle S_i^a S_i^b \rangle} = \int \prod_i \mathscr{D}\hat{h}_i \mathscr{D}h_i \, \overline{e^{-S[\hat{h},h]}} \sum_i S_i^a S_i^b$$

$$= \int \mathscr{D}\hat{C}\mathscr{D}C\mathscr{D}\hat{G}\mathscr{D}G \, e^{-N\mathcal{S}[\hat{C},C,\hat{G},G;0,0]} \sum_i S_i^a S_i^b$$

$$= \int \mathscr{D}\hat{C}\mathscr{D}C\mathscr{D}\hat{G}\mathscr{D}G \left[ \frac{\delta}{\delta i\hat{C}^{ab}} + NC^{ab} \right] e^{-N\mathcal{S}\hat{C},C,\hat{G},G;0,0]}$$

→ *surface term*

$$\frac{1}{N}\sum_i \overline{\langle S_i^a S_i^b \rangle} = \int \mathscr{D}\hat{C}\mathscr{D}C\mathscr{D}\hat{G}\mathscr{D}G \, e^{-N\mathcal{S}[\hat{C},C,\hat{G},G;0,0]} \, C^{ab} = \langle C^{ab} \rangle$$

Average in the $[\hat{C}, C, \hat{G}, G]$ - Field Theory

# Mean Field Limit $N \gg 1$

- In the $N \gg 1$ we can use peak integration $\mapsto$ saddle point

$$\overline{Z}[\hat{b}, b] = \int \mathscr{D}\hat{C}\mathscr{D}C\mathscr{D}\hat{G}\mathscr{D}G\, e^{-N\mathcal{S}[\hat{C},C,\hat{G},G;\hat{b},b]} \sim e^{-N\mathcal{S}_0[\hat{C},C,\hat{G},G;\hat{b},b]} \qquad N \gg 1$$

where $\{\hat{C}, C, \hat{G}, H\}$ follows from the stationary point:

*Self-consistent equations*

$$\frac{\delta}{\delta i\hat{C}^{ab}}\mathcal{S}[(\,\cdot\,)] = 0 \longrightarrow C^{ab} = \langle S^a S^b \rangle_0$$

$$\frac{\delta}{\delta C^{ab}}\mathcal{S}[(\,\cdot\,)] = 0 \longrightarrow i\hat{C}^{ab} = \langle i\hat{h}^a i\hat{h}^b \rangle_0 = 0$$

$$\frac{\delta}{\delta i\hat{G}^{ab}}\mathcal{S}[(\,\cdot\,)] = 0 \longrightarrow G^{ab} = \langle S^a i\hat{h}^b \rangle_0$$

$$\frac{\delta}{\delta G^{ab}}\mathcal{S}[(\,\cdot\,)] = 0 \longrightarrow i\hat{G}^{ab} = \gamma\langle i\hat{h}^a S^b \rangle_0 = \gamma G^{ba}$$

# Mean Field Limit $N \gg 1$

- In the $N \gg 1$ we can use peak integration $\mapsto$ saddle point

$$\overline{Z}[\hat{b}, b] \sim \overline{Z}^{(sp)}[\hat{b}, b] = \left[\int \mathcal{D}\hat{h}\mathcal{D}h\, e^{-L[\hat{h},h;(\cdot)_0]}\right]^N \qquad N \gg 1$$

$$L[\hat{h}, h; (\,\cdot\,)_0] = -\sum_a i\hat{h}^a \left[(1 + \partial_a)\, h^a - \gamma \sum_b G^{ab} S^b - b^a\right] + \frac{1}{2}\sum_{ab} C^{ab} i\hat{h}^a i\hat{h}^b$$

*Self-consistent dynamics of one variable*

# Mean Field Limit $N \gg 1$

- In the $N \gg 1$ limit the dynamics can be reduced to a self-consistent dynamics of a single variables

$$\overline{Z}[\hat{b}, b] \sim \overline{Z}^{(sp)}[\hat{b}, b] = \left[\int \mathscr{D}\hat{h}\mathscr{D}h\, e^{-L[\hat{h},h;(\cdot)_0]}\right]^N \qquad N \gg 1$$

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Use:

$$e^{\frac{1}{2}\sum_{ab} i\hat{h}^a C^{ab} i\hat{h}^b} = \left\langle e^{\sum_a i\hat{h}^a \eta^a} \right\rangle_\eta$$

$\eta$ Gaussian field with

$$\langle \eta^a \eta^b \rangle_\eta = C^{ab}$$

$$\langle \eta^a \rangle_\eta = 0$$

*Self-consistent dynamics of one variable*

$$\longrightarrow \quad \int \mathscr{D}\hat{h}\mathscr{D}h\, e^{-\sum_a i\hat{h}^a[\cdots] + \frac{1}{2}\sum_{ab} C^{ab} i\hat{h}^a i\hat{h}^b} = \left\langle \int \mathscr{D}\hat{h}\mathscr{D}h\, e^{-\sum_a i\hat{h}^a[\cdots - \eta^a]} \right\rangle_\eta$$

# Mean Field Limit $N \gg 1$

- In the $N \gg 1$ limit the dynamics can be reduced to a self-consistent dynamics of a single variables

$\longrightarrow$

$$\partial_a h^a = - h^a + \gamma \sum_b G^{ab} S^b + b^a + \eta^a + h^0 \delta(t_a - t_0)$$

$\eta$ zero-mean Gaussian field with $\langle \eta^a \eta^b \rangle = C^{ab}$

*Stochastic dynamics of one variable*

Note:

$$C^{ab} = \langle S(h^a)\, S(h^b) \rangle_0 = \frac{1}{N} \sum_i \overline{\langle S^a S^b \rangle}$$

$$G^{ab} = \frac{\delta}{\delta b^b} \langle S(h^a) \rangle_0 |_{b=0} = \frac{1}{N} \sum_i \frac{\delta}{\delta b^b} \overline{\langle S^a \rangle} |_{b=0}$$

Self-Consistent Equation

# Mean Field Limit $N \gg 1$

- In the $N \gg 1$ limit the dynamics can be reduced to a self-consistent dynamics of a single variables

$$\partial_a h^a = -h^a + \gamma \sum_b G^{ab} S^b + b^a + \eta^a + h^0 \delta(t_a - t_0)$$

$\eta$ zero-mean Gaussian field with $\langle \eta^a \eta^b \rangle = C^{ab}$

*Stochastic dynamics of one variable*

Note:

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Self-Consistent Equation

Feed-back from symmetric component of synaptic matrix $J_{ij}$

From now on : $\gamma = 0$

→ Fully Asymmetric Matrix

# Self-Consistent Dynamics

$$\longrightarrow \quad \partial_a h^a = -h^a + \eta^a$$

$\eta^a :=$ *Gaussian field of zero mean and correlation* $\langle \eta^a \eta^b \rangle_\eta = C^{ab}$

Where:

$$C^{ab} = \langle S^a S^b \rangle_0 = \frac{1}{N} \sum_i \overline{\langle S_i(t_a)\, S_i(t_b) \rangle}$$

Self-Consistent Equation

$$C(t) = \big\langle S(t+t_0)\, S(t_0) \big\rangle = \Big\langle \phi\big(gh(t+t_0)\big)\, \phi\big(gh(t_0)\big) \Big\rangle$$

$$h(t) = \int_{-\infty}^{t} ds\, e^{-(t-s)}\, \eta(s)$$

*Linear function of* $\eta$

# Differential Self-Consistent Equation

Define

$$\Delta^{ab} = \big\langle h(t_a)\, h(t_b) \big\rangle \equiv \langle h^a h^b \rangle$$ *field-field correlation function*

Self-Consistent dynamics

$$(1 + \partial_a)\, h^a = \eta^a \longrightarrow (1 + \partial_a)(1 + \partial_b)\langle h^a h^b \rangle = \langle \eta^a \eta^b \rangle$$

Taking $t = t_a - t_b$ leads to the differential *self-consistent equation*

$$\rightarrow \quad \boxed{\Delta(t) - \partial_t^2 \Delta(t) = C(t)}$$

*Closed Equation*

$$C(t) \equiv C\big(\Delta(t)\big)$$

# Correlation $C^{ab} = \langle S^a S^b \rangle = \left\langle \phi(gh^a)\, \phi(gh^b) \right\rangle$

*Fourier Transform*

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \tilde{\phi}(k)\, e^{-ikx}$$

$$\longrightarrow \qquad C^{ab} = \int \frac{dk}{2\pi}\frac{dk'}{2\pi}\; \tilde{\phi}(k)\, \tilde{\phi}(k') \left\langle e^{-ikgh^a - ik'gh^b} \right\rangle_\eta$$

$h[\eta]$ linear function of $\eta$ $\longrightarrow$ $h$ *Gaussian Field with* $\langle h^a \rangle = 0, \quad \langle h^a h^b \rangle = \Delta^{ab}$

$$C^{ab} = \int \frac{dk}{2\pi}\frac{dk'}{2\pi}\; \tilde{\phi}(k)\, \tilde{\phi}(k') \exp\left\{ -\frac{g^2}{2}\left[ \Delta_0 (k^2 + k'^2) + 2\Delta k k' \right] \right\}$$

*where*

$$\Delta^{aa} = \Delta^{bb} = \Delta_0 \qquad \Delta^{ab} = \Delta$$

# Self-Consistent Newton Equation

Since $C^{ab} \equiv C(\Delta;\Delta_0)$

$$\partial_t^2\Delta = \Delta - C \equiv -\partial_\Delta V(\Delta;\Delta_0)$$

*Newton Equation*

$\longrightarrow$

$$V(\Delta;\Delta_0) = -\frac{\Delta^2}{2} + \int_0^\Delta d\Delta'\, C(\Delta';\Delta_0)$$

*Classical 1D motion*

Classical Motion with Energy $E_c = \frac{1}{2}(\partial_t\Delta)^2 + V(\Delta;\Delta_0)$

Boundary Conditions

a) $\Delta(t)$ Autocorrelation function $\longrightarrow$ $|\Delta(t)| \le \Delta(t=0) = \Delta_0$ *Bounded Classical Orbits*

b) $\Delta(t) = \frac{1}{2}\int_{-\infty}^{+\infty} dt'\, e^{-|t-t'|}\, C(t')$ $\longrightarrow$ *Differentiable even function of t*

$$\Delta(-t) = \Delta(t)$$

$$\partial_t\Delta(t)|_{t=0} = 0$$

c) $\Delta(t=0) = \Delta_0$ free $\longrightarrow$ *Different solutions with different classical energy* $E_c = V(\Delta_0;\Delta_0)$

# Classical Potential

Noticing:

$$\int_0^{\Delta} d\Delta' \, C(\Delta';\Delta_0) = -\frac{1}{g^2}\int \frac{dk}{2\pi k}\frac{dk'}{2\pi k'}\,\tilde{\phi}(k)\,\tilde{\phi}(k')\exp\left\{-\frac{g^2}{2}\left[\Delta_0(k^2+k'^2)+2\Delta' kk'\right]\right\}\Bigg|_0^{\Delta}$$

Defining:

$$\Phi(x) = \int_0^x dy\,\phi(y) \qquad\qquad \tilde{\Phi}(k) \Rightarrow \tilde{\phi}(k)/k$$

e.g.

$$\phi(x) = \tanh(x) \quad \Rightarrow \quad \Phi(x) = \ln\cos(x)$$

$$\Downarrow$$

$$V(\Delta;\Delta_0) = -\frac{\Delta^2}{2} + \frac{1}{g^2}\int Dz\left[\int Dx\,\Phi\left(gx\sqrt{\Delta_0-|\Delta|}+gz\sqrt{|\Delta|}\right)\right]^2 - \frac{1}{g^2}\left[\int Dx\,\Phi\left(gx\sqrt{\Delta_0}\right)\right]^2$$

where: $Dx = \frac{dx}{\sqrt{2\pi}}e^{-x^2/2}$ *Gaussian Measure*

# Solutions

*) $\frac{\partial^3}{\partial \Delta^3} V(\Delta;\Delta_0) = g^4 \int Dz \left[ \int Dx\, \phi''\left(gx\sqrt{\Delta_0 - |\Delta|} + gz\sqrt{|\Delta|}\right)\right]^2 \quad > 0$

$\longrightarrow$ $\frac{\partial^2}{\partial \Delta^2} V(\Delta;\Delta_0)$ *monotonously increasing function of* $|\Delta|$

*has no zero or has one zero for* $0 < \Delta < \Delta_0$

*) $V(\Delta;\Delta_0) = \left[-1 + g^2 \left[\phi'\right]^2_{\Delta_0}\right] \frac{\Delta^2}{2} + O(\Delta^4) \qquad |\Delta| \ll 1$

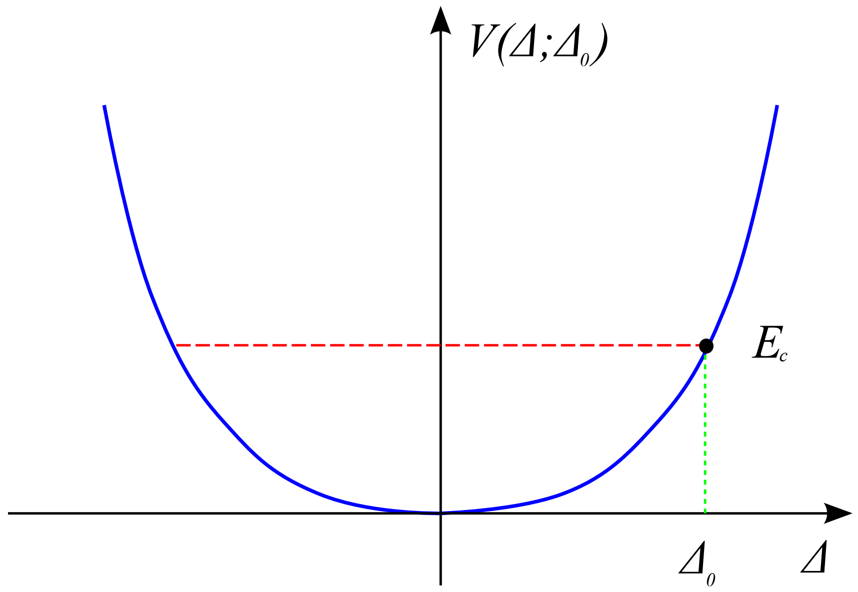
$$[f]_{\Delta_0} = \int Dx\, f\left(gx\sqrt{\Delta_0}\right)$$

Cases:

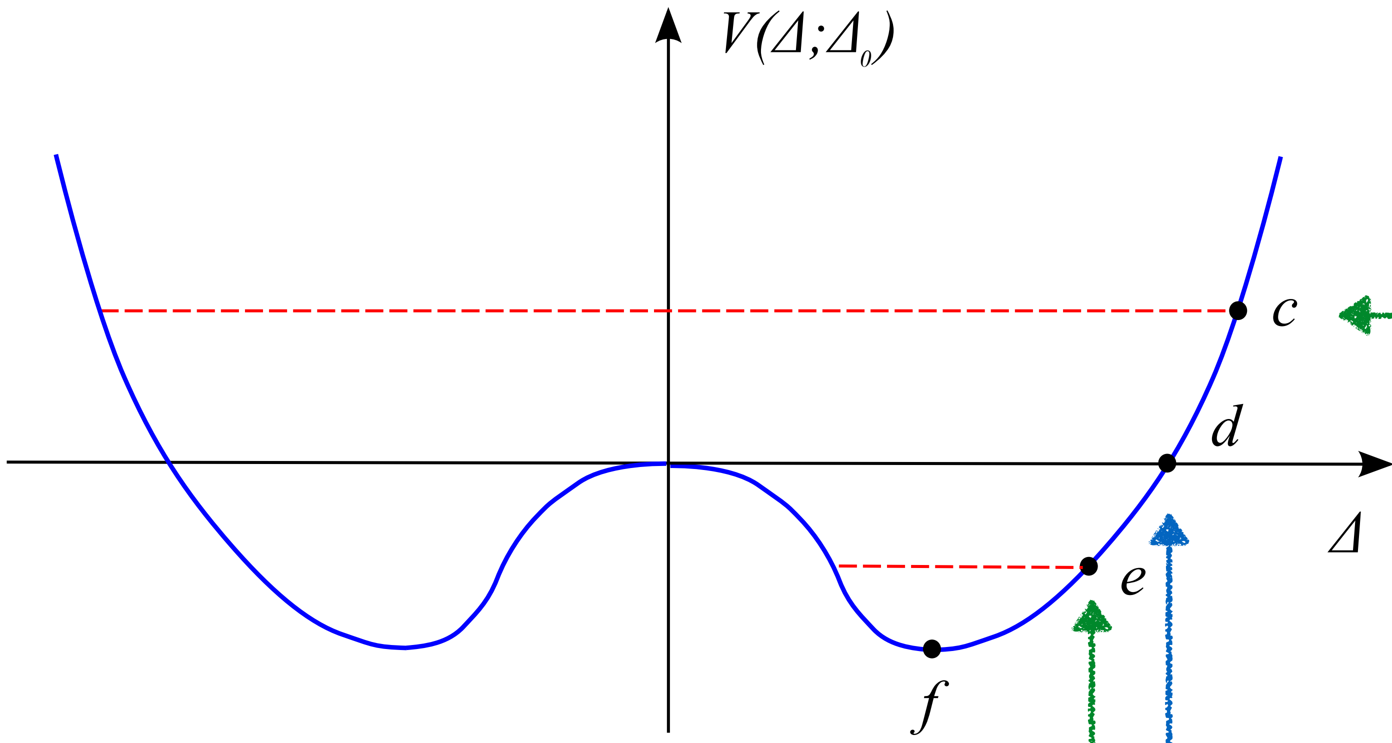
a: $-1 + g^2 \left[\phi'\right]^2_{\Delta_0} > 0$

$E_c = V(\Delta_0;\Delta_0) > 0$

*Oscillating solutions which changes sign*



# Solutions




$$-1 + g^2 \left[\phi'\right]^2_{\Delta_0} < 0$$

$$E_c = V(\Delta_0;\Delta_0)$$

f: $V(\Delta_0;\Delta_0) < 0$

*Time-independent* solution

$$\Delta = \Delta_0 > 0$$

c: $V(\Delta_0;\Delta_0) > 0$

*Oscillating solutions which change sign*

d: $V(\Delta_0;\Delta_0) = 0$

*Time-Decaying solution (chaotic)*

e: $V(\Delta_0;\Delta_0) < 0$

*Oscillating solutions which do not change sign*

Note: for $g < 1$ only f with

$$\Delta = \Delta_0 = 0$$

# Phase diagram

→ $g < 1$

Time-Independent solution

$$\Delta = \Delta_0 = 0$$

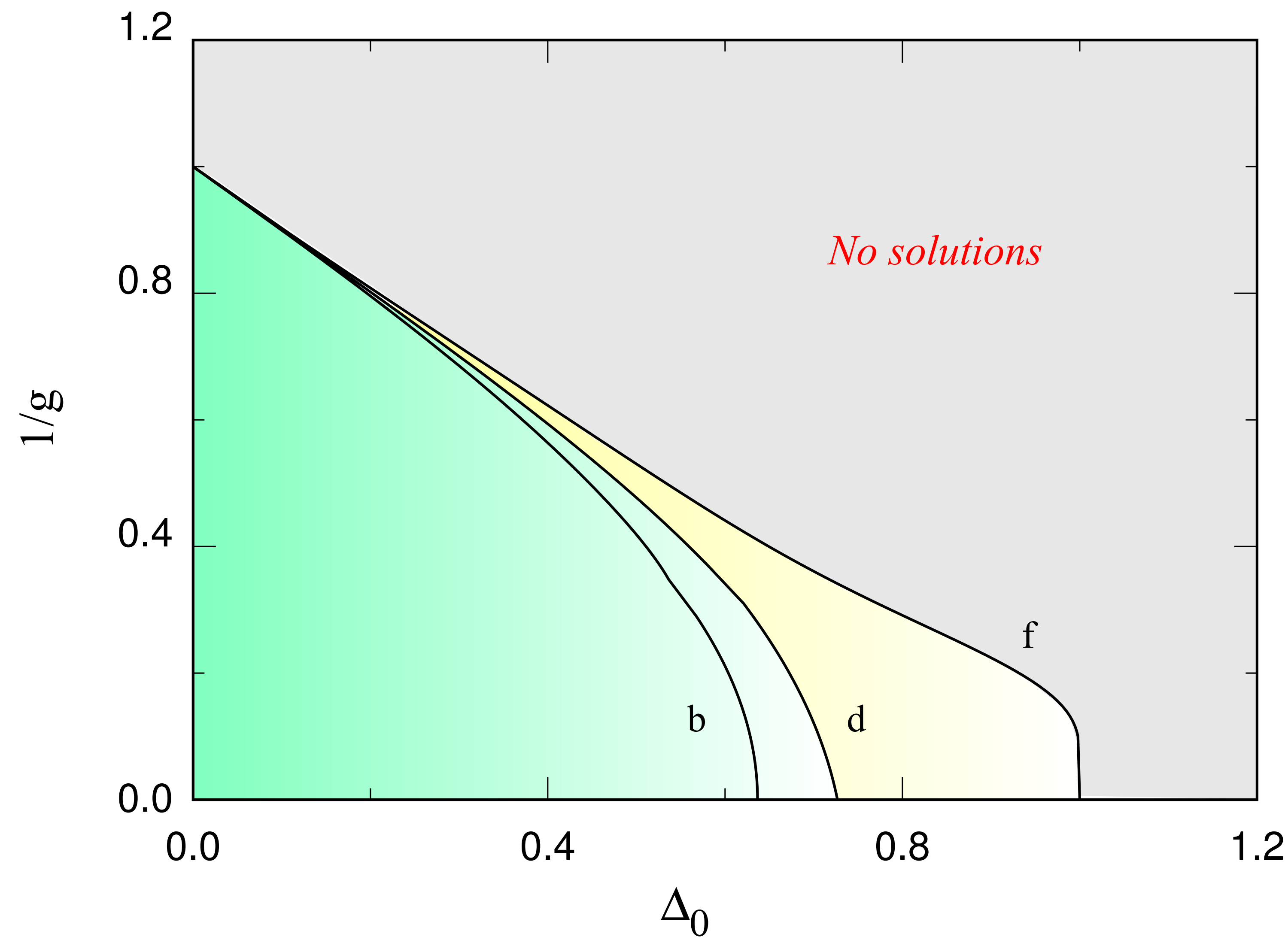
→ $g > 1$

f: Time-Independent solution with

$$\Delta = \Delta_0 = q > 0$$

$$q = [\phi^2]_q = \int Dz\, \phi^2(gz\sqrt{q})$$

*(SK Solution)*




b) $-1 + g^2 \left[\phi'\right]^2_{\Delta_0} = 0$

*single / double well*

f:- d: Oscillating solution which do not change sign

d: Time-decaying solution: $\lim_{t\to\infty} \Delta(t) = 0$

Below d: Oscillating solutions which change sign

# Stability Mean Field Limit

→ Replicas: $S^a \equiv S^a(t_a), \qquad a = 1, \ldots, n$

$$\overline{Z}[\rho,\hat\rho] = \int \mathcal{D}C\,\mathcal{D}\hat C\, e^{-N\mathcal{S}[C,\hat C;\rho,\hat\rho]}$$

$$\mathcal{S}[C,\hat C;\rho,\hat\rho] = \frac{1}{2}\sum_{ab} i\hat C^{ab}C^{ab} - W[C,\hat C;\rho,\hat\rho]$$

*(recall…)*

$$W[C,\hat C;\rho,\hat\rho] = \ln \int \mathcal{D}h\,\mathcal{D}\hat h\, e^{L(h,\hat h;C,\hat C,\rho,\hat\rho)}$$

$$L(h,\hat h;C,\hat C,\rho,\hat\rho) = -\sum_a i\hat h^a(1+\partial_a)h^a + \frac{1}{2}\sum_{ab}\Big[i\hat C^{ab}S^aS^b + C^{ab}i\hat h^a i\hat h^b\Big] + \sum_a \Big[i\hat h^a\rho^a + i\hat\rho^a h^a\Big]$$

→ Fluctuations:

$$C = C^{(sp)} + Q \qquad\qquad i\hat C = i\hat C^{(sp)} + i\hat Q$$

$$\overline{Z}[\rho,\hat\rho] \sim \overline{Z}^{(sp)}[\rho,\hat\rho]\int \mathcal{D}Q\,\mathcal{D}\hat Q\, e^{-N\mathcal{S}_2[Q,\hat Q;\rho,\hat\rho]}$$

$$\mathcal{S}_2[Q,\hat Q;\rho,\hat\rho] = \frac{1}{2}\sum_{ab} i\hat Q^{ab}Q^{ab} - \frac{1}{4}\sum_{ab,cd} i\hat Q^{ab}\langle S^aS^b i\hat h^c i\hat h^d\rangle Q^{cd} - \frac{1}{8}\sum_{ab,cd} i\hat Q^{ab}\Big[\langle S^aS^bS^cS^d\rangle - \langle S^aS^b\rangle\langle S^cS^d\rangle\Big] i\hat Q^{cd}$$

# Stability Mean Field Limit

To evaluate $\langle S^a S^b i\hat{h}^c i\hat{h}^d \rangle$

Define: $$(1+\partial_a)(1+\partial_b)\Psi^{ab} = Q^{ab}$$

$$\longrightarrow \quad \mathcal{S}_2[\Psi,\hat{Q};\rho,\hat{\rho}] = \frac{1}{8}\sum_{ab,cd} i\hat{Q}^{ab}\Big[\langle S^a S^b S^c S^d\rangle - \langle S^a S^b\rangle\langle S^c S^d\rangle\Big] i\hat{Q}^{cd} + \frac{1}{2}\sum_{ab} i\hat{Q}^{ab}\mathcal{A}\Psi^{ab}$$

$$\mathcal{A}\Psi^{ab} = (1+\partial_a)(1+\partial_b)\Psi^{ab} - \frac{\partial}{\partial\Delta^{ab}}\langle S^a S^b\rangle\Psi^{ab} - \frac{\partial}{\partial\Delta^{aa}}\langle S^a S^b\rangle\Psi^{aa} - \frac{\partial}{\partial\Delta^{bb}}\langle S^a S^b\rangle\Psi^{bb}$$

$\hat{Q}$-integration $\longrightarrow$ $$\int \mathcal{D}\Psi \exp\left[-(\cdots)\Psi\mathcal{A}^\dagger\mathcal{A}\Psi\right]$$

Stability Condition:

$$\mathcal{A}\Psi^{ab} = \Lambda\Psi^{ab}, \qquad \Lambda \neq 0$$

# Useful Relations

Define:

$$\xi = x\sqrt{\Delta_0 - |\Delta|} + z\sqrt{|\Delta|}$$

$$\longrightarrow \quad \frac{\partial}{\partial \Delta^{ab}} \langle S^a S^b \rangle = g^2 \int Dz \left[ \int Dx \, \phi'(g\,\xi) \right]^2$$

$$\longrightarrow \quad \frac{\partial}{\partial \Delta^{aa}} \langle S^a S^b \rangle = \frac{g^2}{2} \int Dz \left[ \int Dx \, \phi''(g\,\xi) \right] \left[ \int Dx \, \phi(g\,\xi) \right]$$

# Stability: Time-Independent Solution

Stability Equation:

$$\left[(1+\partial_a)(1+\partial_b) - g^2[(\phi')^2]_q\right]\Psi^{ab} - \frac{g^2}{2}[\phi\phi'']_q\left(\Psi^{aa} + \Psi^{bb}\right) = \Lambda\Psi^{ab}$$

$$\Delta = \Delta_0 = q$$

Case $g < 1$

$$q = 0 \longrightarrow \phi(0) = \phi''(0) = 0, \quad \phi'(0) = 1$$

*Set* $t_a - t_b = t$

*Fourier Transform* $\tilde{f}(\omega) = \int dt\, e^{i\omega t} f(t)$

$$\longrightarrow \quad 1 + \omega^2 - g^2 = \Lambda$$ *Stability Equation*

$\Lambda > 0$ for all $\omega$ $\longrightarrow$ $\Delta = \Delta_0 = 0$ *Stable for* $g < 1$

# Stability: Time-Independent Solution

Case $g > 1$ $\quad q = [\phi^2]_q > 0$

Fluctuations $\quad \Psi^{ba} = \Psi^{ab}$

*Symmetric*

$$\Psi^{ab}(t_a, t_b) = \Psi_S(t_a, t_b)$$

$$\Psi_S(t_b, t_a) = \Psi_S(t_a, t_b)$$

*Anti-Symmetric*

$$\Psi^{ab}(t_a, t_b) = \epsilon^{ab}\Psi_A(t_a, t_b)$$

$$\Psi_A(t_b, t_a) = -\Psi_A(t_a, t_b)$$

$$\epsilon^{ba} = -\epsilon^{ab}$$

*Set* $\quad t_a - t_b = t$

$\longrightarrow \quad 1 + \omega^2 - g^2[(\phi')^2]_q = \Lambda \qquad$ *Relevant Eigenvalue*

*But* $\quad 1 - g^2[(\phi')^2]_q < 0 \qquad$ *[de Almeida Thouless]*

$\exists\ \omega$ such that $\Lambda = 0$ $\quad \longrightarrow \quad \Delta = \Delta_0 = q > 0$ *unstable for* $g > 1$

# Stability: Time-Dependent Solutions

Stability Equation:

$$\left[\partial_a + \partial_b + \partial_a\partial_b - \frac{\partial^2}{\partial\Delta^2}V(\Delta;\Delta_0)\right]\Psi^{ab} - \frac{1}{2}\frac{\partial^2}{\partial\Delta_0\partial\Delta}V(\Delta;\Delta_0)\left[\Psi^{aa}+\Psi^{bb}\right] = \Lambda\Psi^{ab}$$

Critical Fluctuations

$$\Psi^{aa} = 0, \qquad \Psi^{ab} = \Psi(t_a - t_b)$$

*off-diagonal replica fluctuations*

*Set* $\partial_a \equiv \delta + \partial_{t_a}$ $[\delta \to 0^+]$ *to ensure causality*

$$\longrightarrow \quad \left[-\partial_t^2 - \partial_\Delta^2 V(\Delta;\Delta_0)\right]\Psi(t) = \epsilon\Psi(t)$$

*Stability Equation*

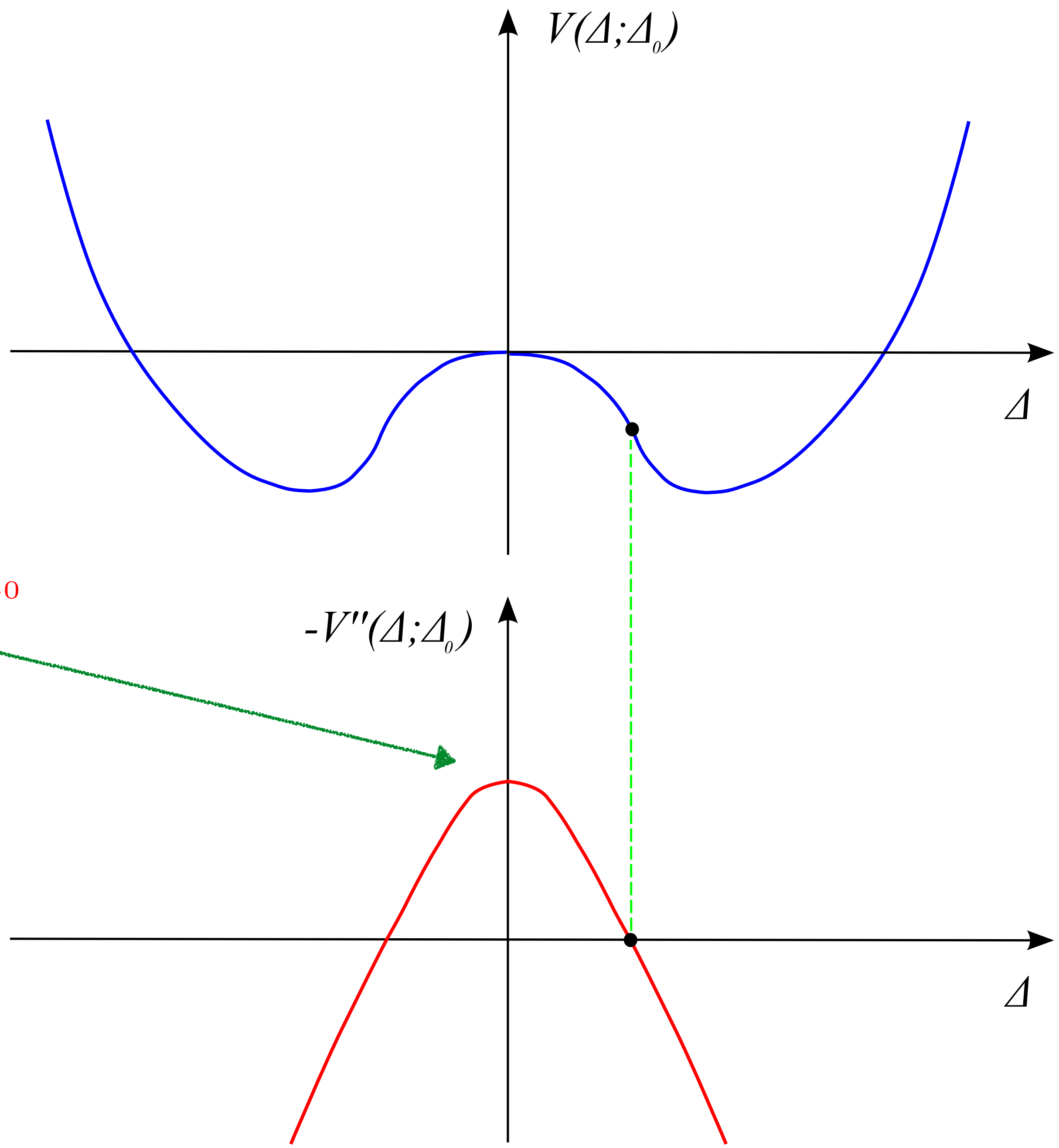
$$\Lambda = 2\delta + \epsilon$$

*1D Schrödinger Equation*

$$V_{\mathrm{QM}}(t) = -\,\partial_\Delta^2 V(\Delta;\Delta_0)\big|_{\Delta=\Delta(t)}$$

# Stability: Time-Dependent Solutions

Classical Potential:




$$-V''(0;\Delta_0) = 1 - g^2[\phi']^2_{\Delta_0}$$

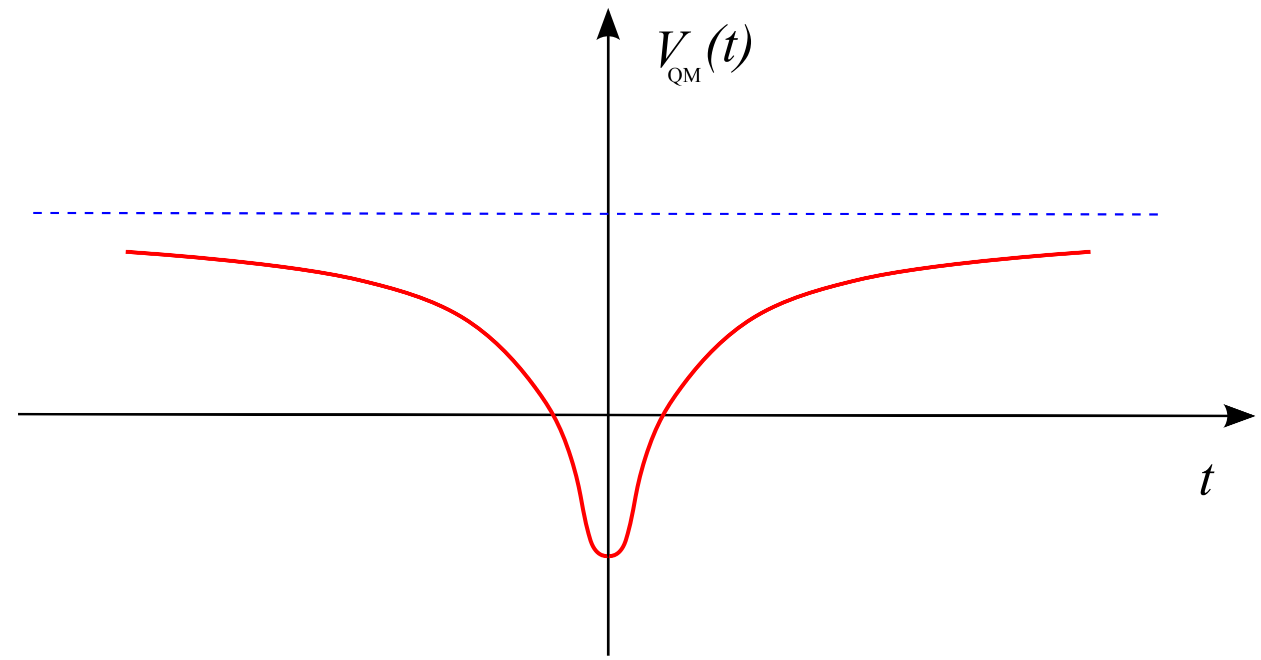
$$\Delta \Rightarrow \Delta(t)$$

# Stability: Time-Dependent Solutions

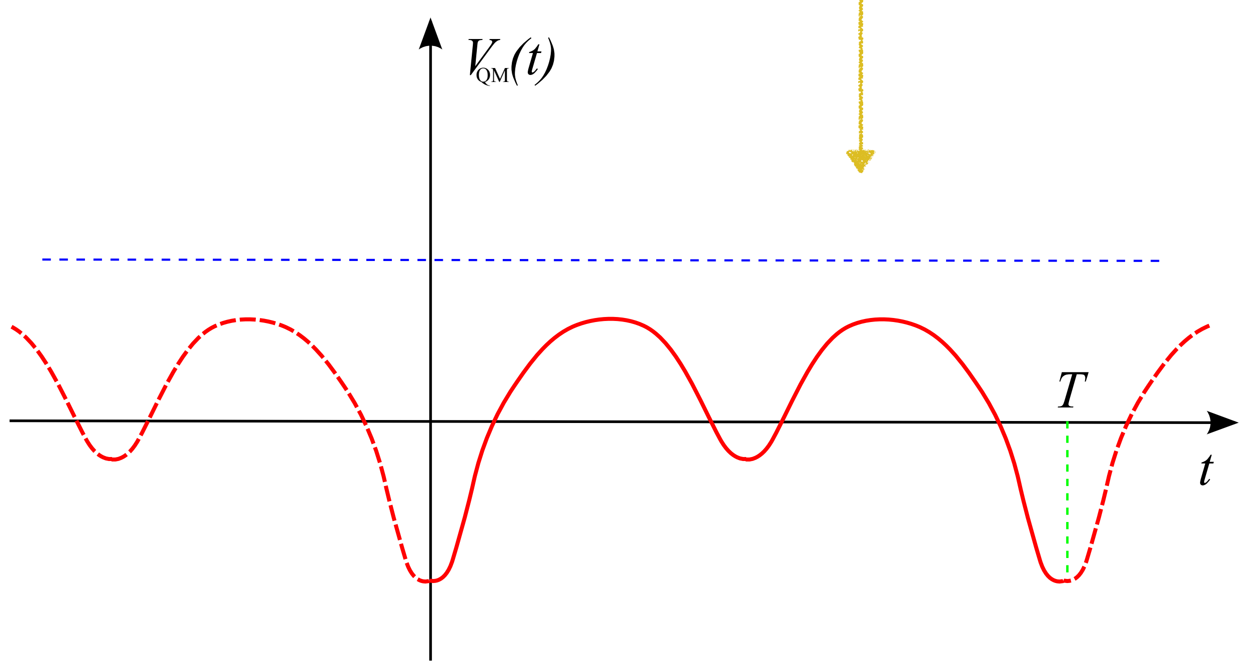
Quantum Mechanical Potential:

$$V_{\mathrm{QM}}(t) = -\,\partial^2_\Delta V(\Delta;\Delta_0)\big|_{\Delta=\Delta(t)}$$

$$-V''(0;\Delta_0) = 1 - g^2[\phi']^2_{\Delta_0}$$




*Time-Decaying solution*




*Time-Periodic solution*

# Stability: Time-Dependent Solutions

Self-Consistent Equation:

$$\partial_t^2 \Delta = -\partial_\Delta V(\Delta;\Delta_0) \longrightarrow \left[-\partial_t^2 - \partial_\Delta^2 V(\Delta;\Delta_0)\right] \partial_t \Delta = 0$$

$\partial_t \Delta$ *eigenvector with eigenvalue* $\epsilon = 0$ $\longrightarrow$ $\Lambda = 2\delta$

*marginally stable* $\delta \to 0^+$

Time-Decaying solution:

$\Psi(0) = \partial_t \Delta(t)|_{t=0} = 0$ $\longrightarrow$ $\Psi = \partial_t \Delta$ *has exactly one node*

*There is exactly one eigenfunction with eigenvalue* $\epsilon_0 < 0$

$$\boxed{\Lambda = \epsilon_0 + 2\delta \neq 0} \longrightarrow \textit{stable}$$

# Stability: Time-Dependent Solutions

Time-Periodic solution:

$\Psi(t)$ periodic solution of period $T$ → $\Psi = \partial_t \Delta$ *changes sign once in a period T and vanishes at t = 0, T*

*There is exactly one periodic eigenfunction with eigenvalue* $\epsilon_0 < 0$

But…..

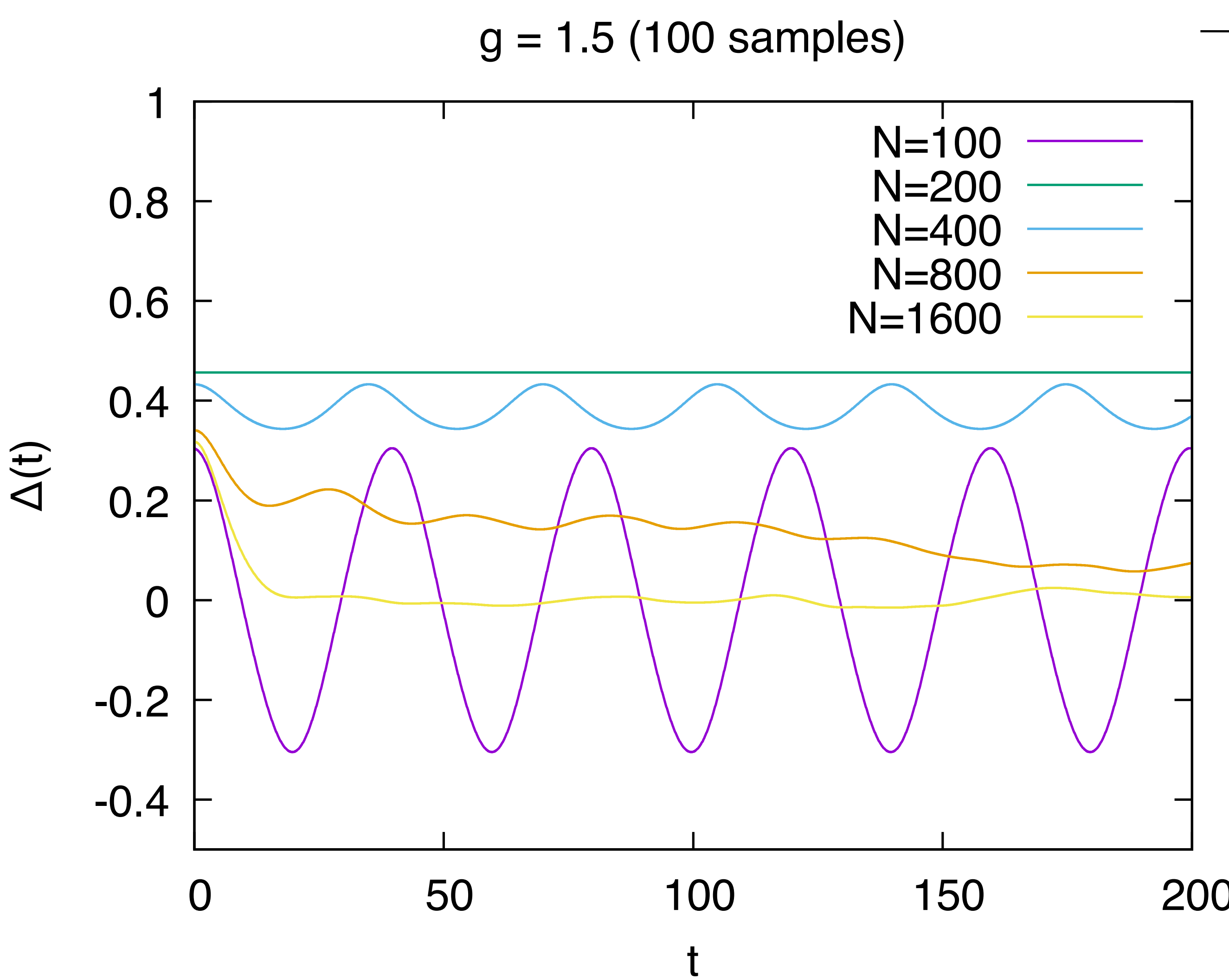
*Periodic potential* → *Energy bands*

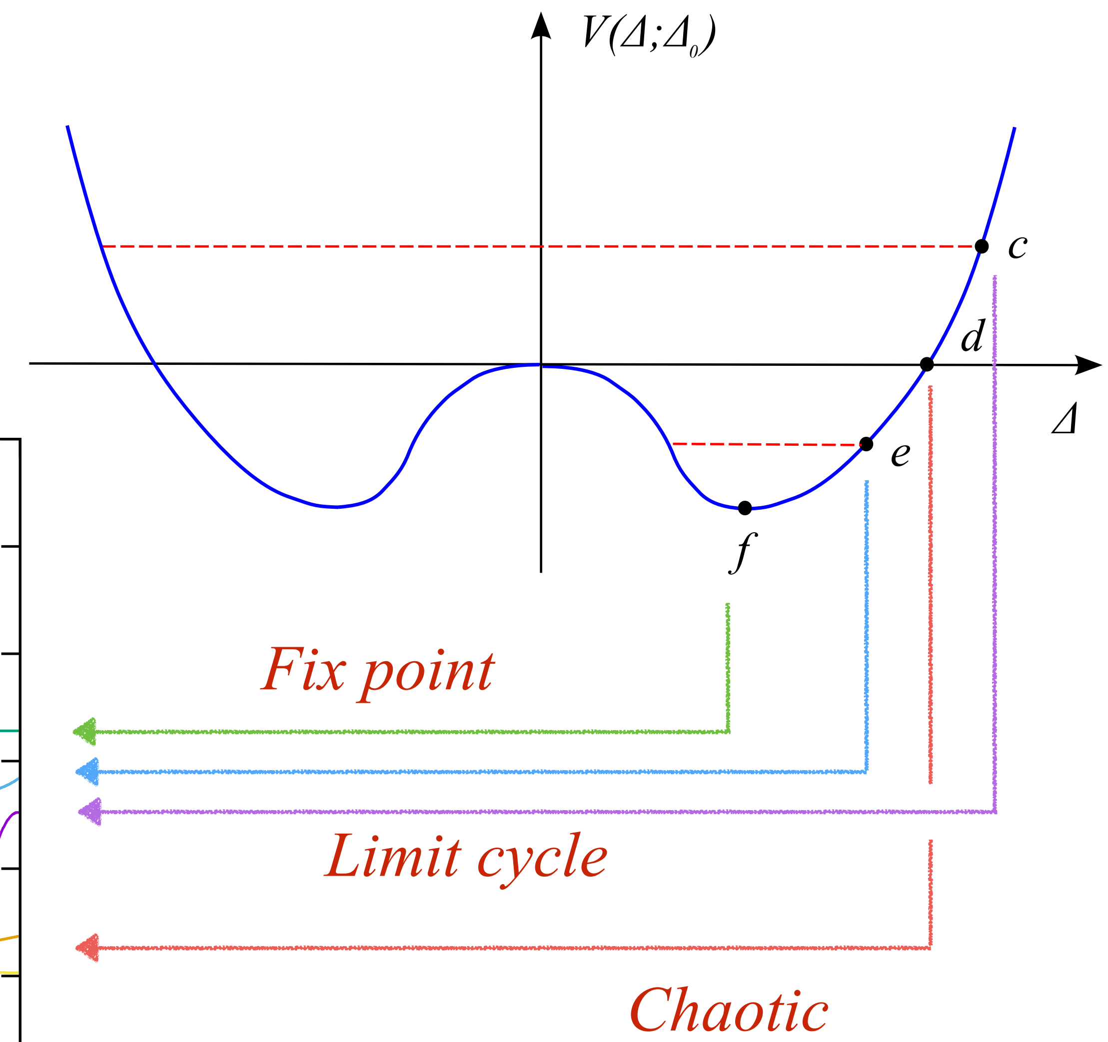
$\epsilon_0 < 0$ *bottom of lowest energy bands*

$\Lambda$ *passes continuously through 0* → *unstable*

# Some Numerical Result

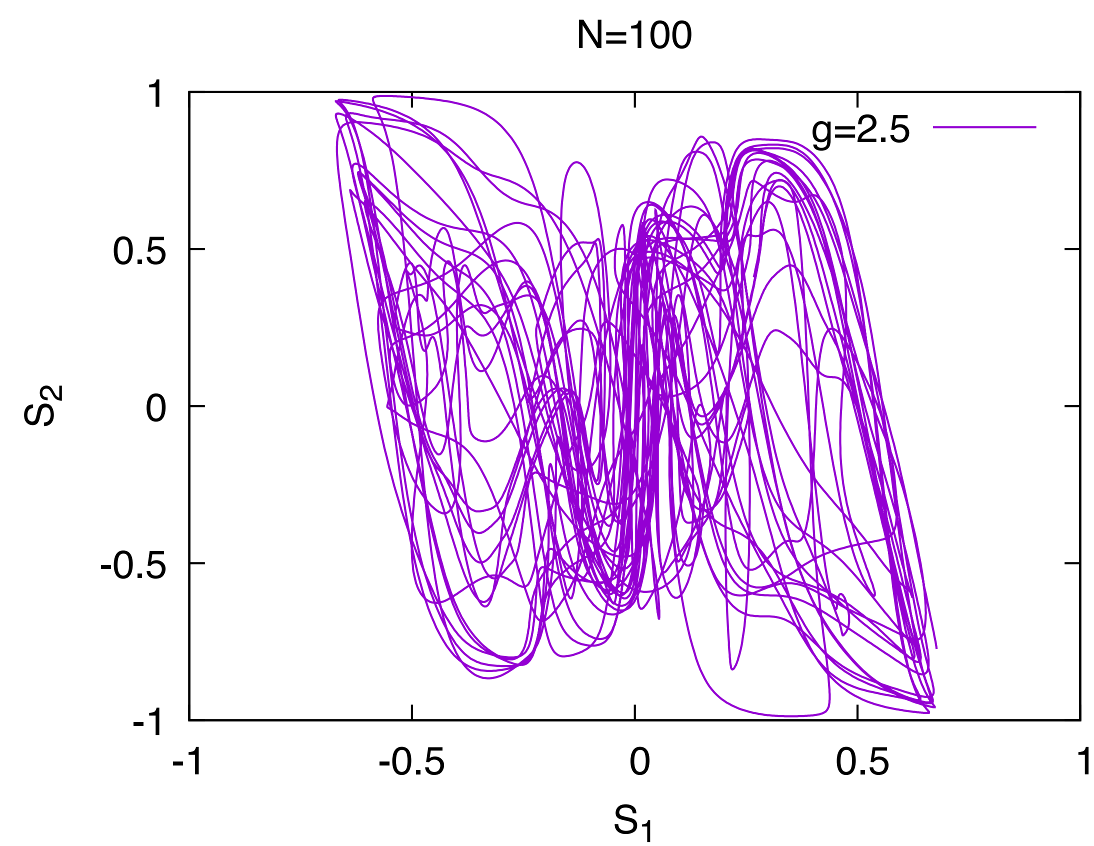
→ Fixed *g*:



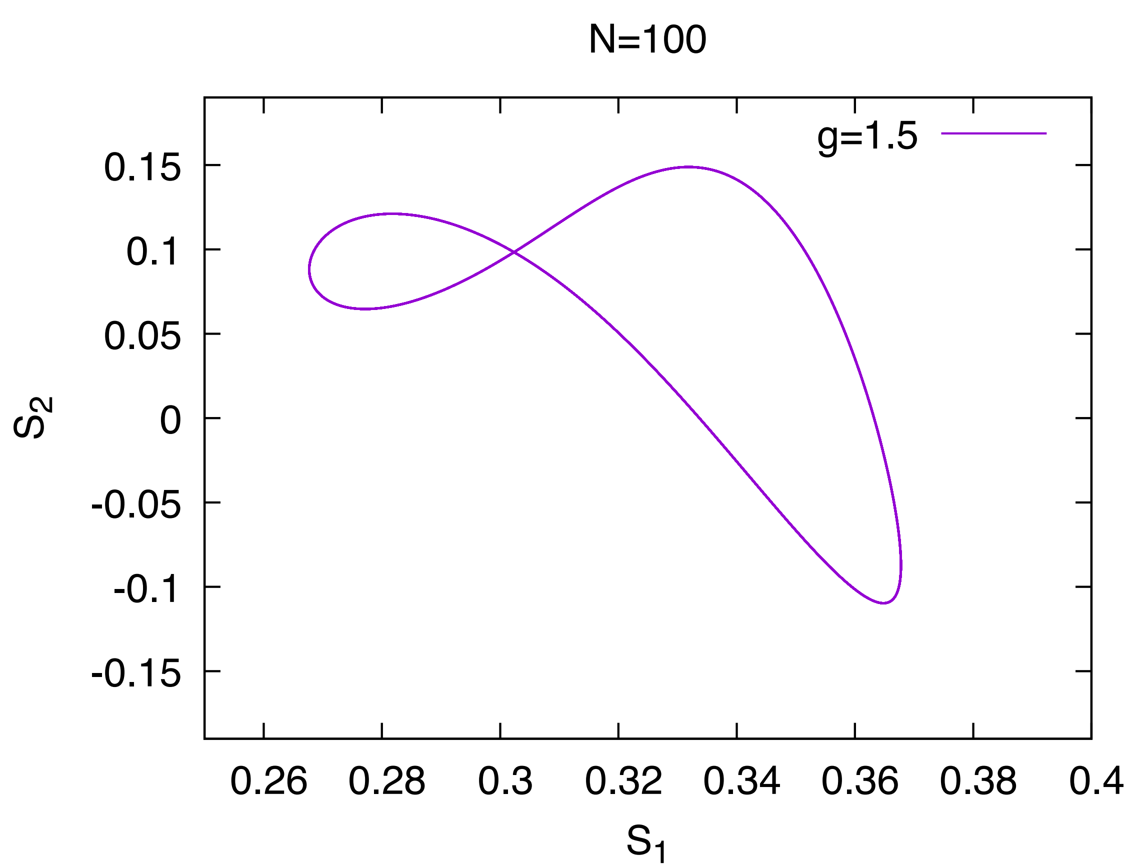



# Some Numerical Result

→ Fixed *N*:




*Chaotic behaviour*



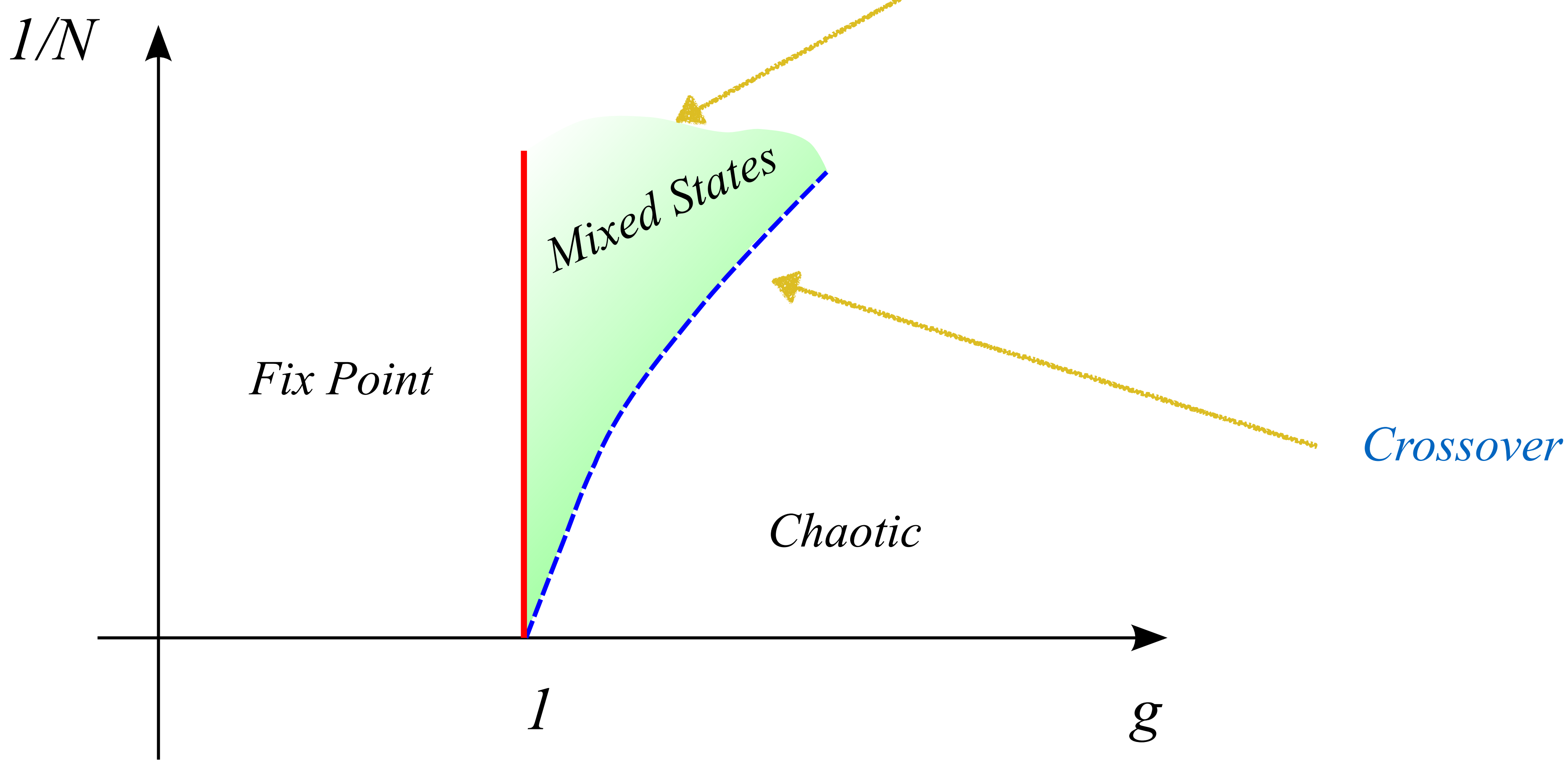

*Periodic behaviour*

# Some Numerical Result

Finite $N$ → Smooth transition *Fix Point* / *Chaotic*

$h_i(0)$ → Behaviour depends on $J_{ij}$

$J_{ij}$ → Behaviour depends on $h_i(0)$



# Maximal Lyapunov Exponent

Lyapunov Exponent controls the growth of small perturbations

$$h_i(t=0) \to h_i(t=0) + \delta h_i(0)$$

$$\longrightarrow \quad |\delta h(t)| \sim |\delta h(0)|\, e^{\lambda t}, \qquad t \gg 1$$

$\lambda :=$ *Lyapunov Exponent*

$\lambda > 0$ *Chaotic Behaviour*

How do we compute $\lambda$ ….

*) Perturbation evolution law:

$$\partial_t \delta h_i = -\delta h_i + g \sum_{j=1}^{N} J_{ij}\, \phi'(g h_j)\, \delta h_j$$

*Linearised equation of motion*

$$\longrightarrow \quad \delta h_i(t) = \sum_{j=1}^{N} \tilde{\chi}_{ij}(t, t_0)\, \delta h_j(t_0)$$

$$\tilde{\chi}_{ij}(t, t_0) = \left. \frac{\delta h_i(t)}{\delta h_j(t_0)} \right|_{h_j = 0}$$

# Maximal Lyapunov Exponent

*) Perturbation growth:

$$\frac{1}{N}\sum_{i=1}^{N}\delta h_i(t)^2 = \sum_{jk} A_{jk}(t,t_0)\,\delta h_j(t_0)\delta h_k(t_0)$$

$$A_{jk}(t,t_0) = \frac{1}{N}\sum_{i=1}^{N}\tilde{\chi}_{ij}(t,t_0)\,\tilde{\chi}_{ik}(t,t_0)$$ *symmetric matrix*

*) Lyapunov Exponent:

$$\frac{1}{N}\sum_{i=1}^{N}\delta h_i(t)^2 \sim e^{2\lambda t}, \qquad t \gg 1$$

$\longrightarrow$

$$\lambda = \lim_{t\to\infty}\frac{1}{2t}\ln\Big[\mathrm{Tr}\,A(t,t_0)\Big] = \lim_{t\to\infty}\frac{1}{2t}\ln\left[\frac{1}{N}\sum_{ij}\tilde{\chi}_{ij}(t,t_0)^2\right]$$

$N \gg 1$ *self-averaging*

# Maximal Lyapunov Exponent

*) Spin Susceptibility:

$$\chi_{ij}(t,t_0) = \frac{\delta S_i(t)}{\delta h_j(t_0)}\Big|_{h_j=0} = g\phi'(gh_i)\,\tilde{\chi}_{ij}(t,t_0)$$

$$\longrightarrow \qquad \lambda = \lim_{t\to\infty} \frac{1}{2t} \ln\left[\frac{1}{N}\sum_{ij}\chi_{ij}(t,t_0)^2\right]$$

*) Dynamic Field Theory:

$$\chi_{ij}(t,t_0) \to \left\langle S_i(t)\, i\hat{h}_j(t_0)\right\rangle$$

*replicas*

$$\longrightarrow \qquad \frac{1}{N}\sum_{ij}\chi_{ij}(t,t_0)^2 \to \frac{1}{N}\sum_{ij}\langle S_i^a S_i^b i\hat{h}_j^c i\hat{h}_j^d\rangle \qquad a = c \neq b = d \qquad \textit{replica indexes}$$

# Maximal Lyapunov Exponent

$$\frac{1}{N}\sum_{ij}\langle S_i^a S_i^b i\hat{h}_j^c i\hat{h}_j^d\rangle = N\langle C^{ab} i\hat{C}^{cd}\rangle - \frac{1}{2N}\left[\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right]$$

$\langle C^{ab} i\hat{C}^{cd}\rangle$ two-point correlation function of the theory governed by the action $N\mathcal{S}[C,\hat{C};0,0]$

where:

$$\mathcal{S}[C,\hat{C};0,0] = \frac{1}{2}\sum_{ab} i\hat{C}^{ab}C^{ab} - W[C,\hat{C};0,0]$$

$$W[C,\hat{C};0,0] = \ln\int \mathcal{D}h\,\mathcal{D}\hat{h}\, e^{L(h,\hat{h};C,\hat{C},0,0)}$$

*dynamic single-site field theory*

$$L(h,\hat{h};C,\hat{C},0,0) = -\sum_a i\hat{h}^a(1+\partial_a)h^a + \frac{1}{2}\sum_{ab}\left[i\hat{C}^{ab}S^aS^b + C^{ab}i\hat{h}^a i\hat{h}^b\right]$$

# Maximal Lyapunov Exponent $N \gg 1$

*Saddle point*

$$\mathcal{S}[C,\hat{C}] \sim \mathcal{S}^{(sp)}[C,0] + \mathcal{S}_2[Q,\hat{Q}], \qquad N \gg 1$$

where: (*see fluctuations*)

$$\mathcal{S}_2[\Psi,\hat{Q}] = \frac{1}{8}\sum_{ab,cd} i\hat{Q}^{ab}\Big[\langle S^a S^b S^c S^d\rangle - \langle S^a S^b\rangle\langle S^c S^d\rangle\Big] i\hat{Q}^{cd} + \frac{1}{2}\sum_{ab} i\hat{Q}^{ab}\mathcal{A}\Psi^{ab}$$

$$\mathcal{A}\Psi^{ab} = (1+\partial_a)(1+\partial_b)\Psi^{ab} - \frac{\partial}{\partial\Delta^{ab}}\langle S^a S^b\rangle\Psi^{ab} - \frac{\partial}{\partial\Delta^{aa}}\langle S^a S^b\rangle\Psi^{aa} - \frac{\partial}{\partial\Delta^{bb}}\langle S^a S^b\rangle\Psi^{bb}$$

$$(1+\partial_a)(1+\partial_b)\Psi^{ab} = Q^{ab}$$

$$\mathcal{S}_2[\Psi,\hat{Q}] = \frac{1}{2}(i\hat{Q},\Psi)\begin{bmatrix} a & b \\ c & 0\end{bmatrix}\begin{pmatrix} i\hat{Q} \\ \Psi\end{pmatrix}$$

$$\mathcal{A}\left\langle\Psi^{ab} i\hat{Q}^{cd}\right\rangle = \frac{1}{2N}\left[\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right]$$

*Equation for* $\frac{1}{N}\sum_{ij}\chi_{ij}(t,t_0)^2$

$$(1+\partial_a)(1+\partial_b)\left\langle\Psi^{ab} i\hat{Q}^{cd}\right\rangle = \left\langle Q^{ab} i\hat{Q}^{cd}\right\rangle$$

# Maximal Lyapunov Exponent $N \gg 1$

*) *Take* $a = c \neq b = d$

$$\Big[\partial_a + \partial_b + \partial_a\partial_b - \partial^2_\Delta V(\Delta;\Delta_0)\Big]\big\langle \Psi^{ab}(t_a,t_b)\, i\hat{Q}^{ab}(t_c,t_d)\big\rangle = \frac{1}{2N}\delta(t_a - t_c)\delta(t_b - t_d)$$

Define:

$$t = t_a + t_b, \quad t' = t_c + t_d$$
$$s = t_a - t_b, \quad s' = t_c - t_d$$

$$\longrightarrow \Big[2\partial_t + \partial_t^2 + \mathcal{H}_s\Big]\big\langle \Psi^{ab}(t,s)\, i\hat{Q}^{ab}(t',s')\big\rangle = \frac{1}{N}\delta(t-t')\delta(s-s')$$

$$\mathcal{H}_s = -\partial_s^2 - \partial^2_\Delta V(\Delta;\Delta_0)\Big|_{\Delta=\Delta(s)}$$

*Fluctuations Quantum Hamiltonian*

Write:

$$\big\langle \Psi^{ab}(t,s)\, i\hat{Q}^{ab}(t',s')\big\rangle = \frac{1}{N}\sum_n g_n(t,t')\,\varphi_n(s)\,\varphi_n^*(s')$$

↓

Hamiltonian eigenfunctions: $\mathcal{H}\,\varphi_n = \epsilon_n\,\varphi_n$

# Maximal Lyapunov Exponent $N \gg 1$

$$\longrightarrow \qquad \left[2\partial_t + \partial_t^2 + \epsilon_n\right] g_n(t,t') = \delta(t-t')$$

Fast:

*Assume* $\quad g_n(t,t') \sim e^{2\lambda_n (t-t')}, \qquad t - t' \gg 1$

$$2\lambda_n = -1 \pm \sqrt{1-\epsilon_n}$$

Slow:

*Exact* $\quad g_n(t-t') = \frac{\theta(t-t')}{\sqrt{1-\epsilon_n}}\, e^{-(t-t')} \sinh\left[\sqrt{1-\epsilon_n}(t-t')\right]$

Finally:

$$\boxed{\lambda = \max_n \lambda_n = \lambda_0 = \frac{1}{2}\left[-1 + \sqrt{1-\epsilon_0}\right]}$$

*Maximal Lyapunov exponent*

$\epsilon_0 :=$ *lowest eigenvalue of* $\mathcal{H}$ $\quad \longrightarrow \quad \epsilon_0 < 0 \quad \Rightarrow \quad \lambda > 0$

# Maximal Lyapunov Exponent $N \gg 1$

$$\longrightarrow \quad V_{QM} = -\partial^2_\Delta V(\Delta;\Delta_0) = 1 - g^2 \int Dz \left[ \int Dx\, \phi'\left( gx\sqrt{\Delta_0 - |\Delta|} + gz\sqrt{|\Delta|} \right) \right]^2$$

*) $g < 1 \quad \Rightarrow \quad \Delta = \Delta_0 = 0$

[ $\phi'(0) = 1$ ] $\quad V_{QM} = 1 - g^2 \quad \longrightarrow \quad \epsilon_0 = 1 - g^2$

$$\lambda = \frac{1}{2}(-1+g) < 0, \qquad g < 1$$

*) $g > 1$ *decaying solution* $\Delta(t) \sim e^{-t/\tau_a}, \quad t \gg 1$

$\longrightarrow \quad \epsilon_0 < 0$ $\quad \lambda > 0 \quad g > 1$ $\quad$ *Chaotic Solution*

*Limits*

$$\lambda \sim \frac{1}{4}(g-1)^2 \qquad g \to 1^+ \qquad \lambda^{-1} \gg \tau_a \sim (g-1)^{-1}$$

$$\lambda \sim C\,\ln g \qquad g \to \infty \qquad \lambda^{-1} \ll \tau_a \sim 1/\sqrt{1-2/\pi} \qquad \textit{(PRM Theory)}$$

$$2C = 1/\sqrt{(\pi-2)(2-\pi/2)} \simeq 1.4286\ldots$$

# Final Remarks

That's enough…

- Dynamic Field Theory Formulation
- Mean Field Limit
  - Exact in this model (fully connected)
    - Finite $K, D$ Corrections → Perturbative expansions
  - Solvable in this model (fully asymmetric)
    - symmetry → Memory effects
- Dynamic Mean Field Theory
  - Starting point perturbative corrections
- Lyapunov exponent

Thanks for your attention

(and patience)

□