The observer with the resting clock finds a larger value for the time interval than that corresponding to the proper time, i.e. than the time interval indicated by the clock moving with the body.

This time dilation is especially impressive in fast-moving atomic particles which, within themselves, carry a "clock" keeping the proper time. This is the case, for example, with radioactive nuclei which, on the average after a certain time τ_0, go over to the product body by the emission of a charged particle. The average lifetime τ_0 is defined as that time in which the number of still-undecayed nuclei has become reduced to $1/e$ of the original number. If such substances are observed in a system with respect to which they move with velocity v, the lifetime undergoes a lengthening by the factor $1/\sqrt{(1-v^2/c^2)}$, corresponding to (77.1). In the radioactive particles of cosmic radiation, in particular in μ-mesons which are produced in the upper regions of the earth's atmosphere by collisions of energetic particles coming from cosmic space, this dilation factor (v being almost equal to c) can reach values approaching 10^4. The consequence is that these μ-mesons, for which $\tau \approx 2 \times 10^{-6}$s, reach the earth's surface in enormous numbers before their decay and are observed—this in spite of their relatively long travel time (approximately 3×10^{-5}s) through about 10km thickness of homogeneous atmosphere.

(*b*) *Geometrical representation of the Lorentz transformation.* These and similar consequences of equations (76.5) become more clear when, as *Minkowski* first showed, the contents of the equations are visualized in a geometrical fashion. We leave out of consideration the y- and

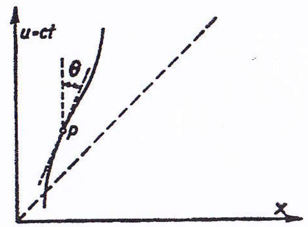

Fig. 66.—World line of a point in the xu-plane. Its velocity is $c\tan\theta$

z-axes since these are not altered by the transformation (76.5) when the motion of both systems is along the x-axis. We represent all possible point events in the first system by a space-time diagram in which an abscissa denotes the location x, and an ordinate denotes $u = ct$, the time multiplied by c. The motion of a material point appears in our scheme (figure 66) as a curve (*world line* of the point), the tangent of which forms with the time axis the angle $\theta = \tan^{-1}(dx/du) = \tan^{-1}(v/c)$, v being the momentary velocity of the point. Since we exclude from consideration any velocity faster than light, the angle of inclination of these curves with the time axis has always to be smaller than 45°. The motion diagram for a light wave is a straight line inclined at 45° with respect to the axis.

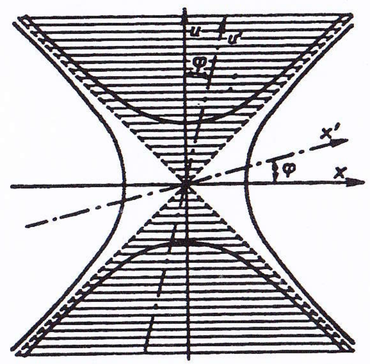

Fig. 67.—Transfer from system (x, u) to the system (x', u'). The hyperbolas intersect the respective axes at unit values of the variables

In addition to the first (unprimed) system (I), we consider (figure 67) a second system (II) moving with respect to the first with velocity v along the x-axis. With the abbreviations $u = ct$, $u' = ct'$, $\beta = v/c$, the equations to be discussed, namely (76.5) and (76.6), read:

$$x' = \frac{x-\beta u}{\sqrt{(1-\beta^2)}}, \qquad u' = \frac{-\beta x+u}{\sqrt{(1-\beta^2)}}$$

$$x = \frac{x'+\beta u'}{\sqrt{(1-\beta^2)}}, \qquad u = \frac{\beta x'+u'}{\sqrt{(1-\beta^2)}} \qquad (77.2).$$

In our scheme there corresponds to system II (itself in motion with respect to I) a coordinate system whose axes we find in the following way: By definition, the points ($x = 0$, $u = 0$) and ($x' = 0$, $u' = 0$) fall together. The point $x' = 0$ is moving with respect to I with velocity v; its world line is therefore a straight line through 0 forming with the time axis the angle $\phi = \tan^{-1}\beta$. Since $x' = 0$ for this line, it is identical with the time axis of the primed system. By setting $u' = 0$ in (77.2) we obtain the space axis, whose equation thus reads $ct = u = \beta x$; this is a straight line which forms the same angle $\phi = \tan^{-1}\beta$ with the x-axis.

From this presentation we recognize with special emphasis the relative character of simultaneity. All point events on the x'-axis appear to the second observer as *simultaneous*, while for the first observer they are *progressive*. For the first observer the point event A' (figure 68) occurs for him later by a time interval $u/c = (1/c)$. AA′ than the event 0.

For a complete representation of the Lorentz transformation we still need the units on the axes. For this purpose we draw in figure 67 the two rectangular hyperbolas

$$x^2 - u^2 = 1 \quad \text{and} \quad u^2 - x^2 = 1 \qquad (77.3)$$

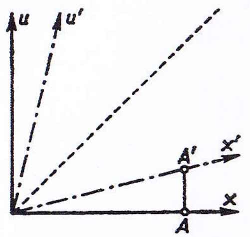

Fig. 68.—For the relativity of simultaneity

They cut the u- and x-axes of the unprimed system at the points $u = 1$, $x = 0$, and $x = 1$, $u = 0$. That they cut the axes of the primed system in the points $(u' = 1, x' = 0)$ and $(u' = 0, x' = 1)$ follows immediately from relation (76.7) with $u = ct$ and $u' = ct'$, to give

$$x^2 - u^2 = x'^2 - u'^2 = \pm 1$$

With the aid of figure 69, the phenomenon of *reciprocal scale-shortening* can be described in the following way: Let OA be a unit scale at rest in the first system. The world lines of its end points are ODC and AA′. For an observer at rest in the second system, the corresponding and simultaneous position of the scale ($u' = 0$) gives the location of its beginning and end points at the world points O and A′ respectively.

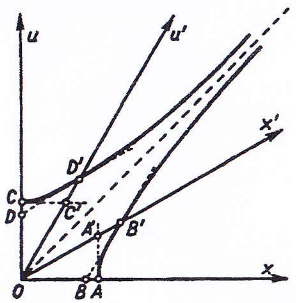

Fig. 69.—For comparing units of scales or clocks in mutual motion

The scale, therefore, is shorter for him than his unit length OB′. Conversely, the beginning and end of a unit length OB′ at rest in the second system with world lines OC′D′ and BB′ appears for the first observer at the time $u = 0$ at the world points O and B. It again appears shorter than the unit scale OA of the resting system.

The comparison of clocks takes place in a completely analogous manner. A clock at rest in the second system moves on the world line OC′D′. At the world point D′ ($u' = 1$) it has completed one revolution; but already before this (at the world point C′) the clock of the first system with which it is in space coincidence has completed a revolution ($u = 1$). The moving clock therefore runs slower than the clock at rest. A clock at rest (at $x = 0$) in the first system has completed a revolution at world point C, while already at D a clock located at the same place in the second system has completed a revolution.

In this way we can be fully and intuitively convinced that the assertion of the reciprocal shortening of scales and the retardation of clocks contains absolutely nothing paradoxical as soon as we have renounced the notion of absolute simultaneity.

(c) *The Einstein addition theorem for velocities.* We wish now, with the aid of the Lorentz transformation, to derive the *Einstein theorem for the addition of two velocities*. According to the old kinematics this consisted simply in the vectorial addition of the velocities: If $\mathbf{v}$ is the velocity of, say, a ship (i.e. a coordinate system) with respect to a certain coordinate system, and if a mass point moves with velocity $\mathbf{u}'$ on the ship, then the mass point moves with velocity $\mathbf{u} = \mathbf{v} + \mathbf{u}'$ relative to an observer at rest.

In relativity theory this relationship is substantially more complicated. We examine further the two coordinate systems connected by equations (76.5) and we assume that a mass point moves relative to the primed system with the velocity u' in the $x'y'$-plane so that its trajectory makes an angle θ' with the x'-axis. Its equation of motion, in the primed system, is then

$$x' = u't'\cos\theta' \qquad y' = u't'\sin\theta' \qquad z' = 0 \qquad (77.4)$$

We wish now to describe the same world line from the point of view of the *first* system, i.e. we wish to determine two quantities u and θ in such a manner that the equations

$$x = ut\cos\theta \qquad y = ut\sin\theta \qquad z = 0 \qquad (77.5)$$