Mappe della radiazione di fondo cosmico

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What a bolometer measures

Let's consider one single bolometer in the focal plane of a scanning telescope.

In every time sample the bolometer measures a convolution of the sky with the beam in the band:

$$V(t) = \mathcal{R} A \int_0^\infty d\nu \int_{4\pi} d\Omega R(\theta - \theta_0, \phi - \phi_0) B(\theta, \phi, \nu) \eta F(\nu) + n$$

Output [Volt]
Responsivity [Volt/W]
Area of the telescope [m^2]
Solid angle [sr]
Angular response
Direction in the sky (pointing)
Brightness of the sky [Jy/sr]
Telescope efficiency
Band transmission
Noise

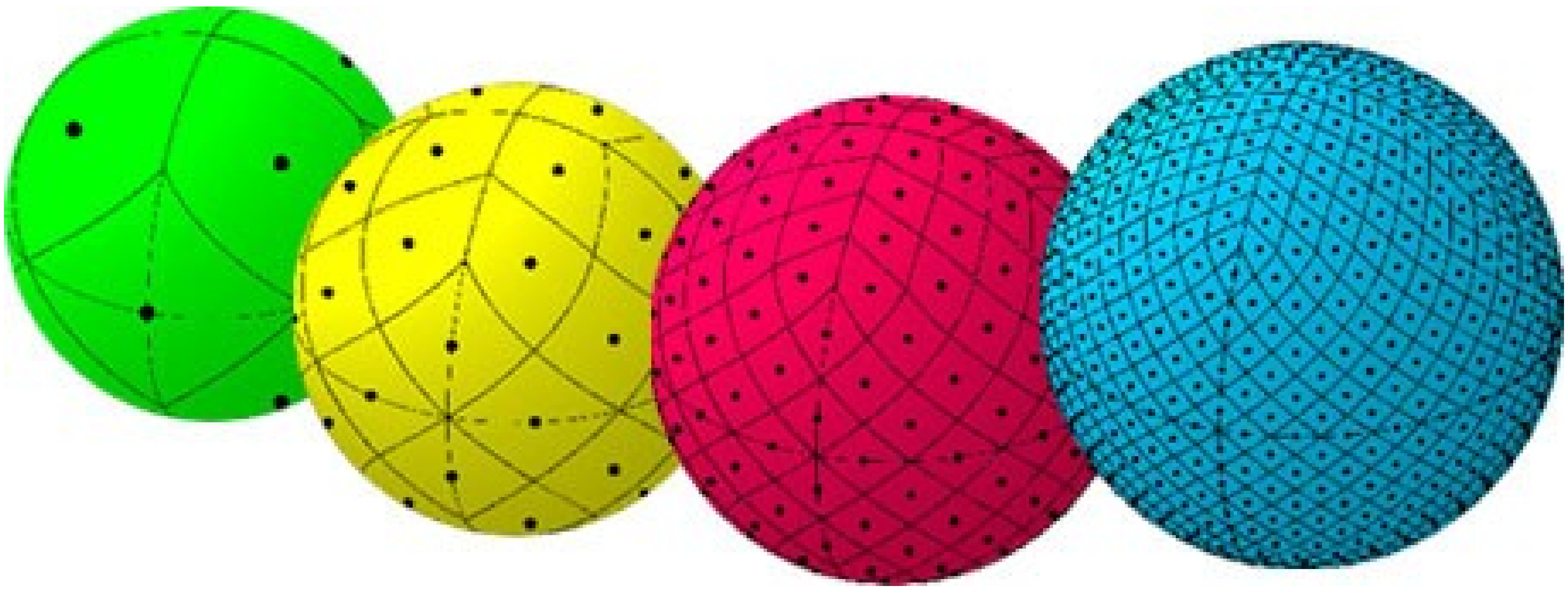
Pixellization

The sky is divided in pixels. To every pixel a pixel number is associated.

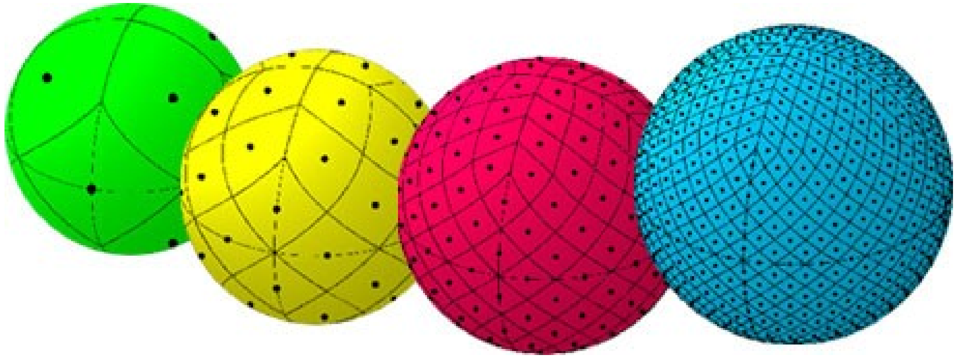
Every direction in sky fall in a pixel.

Pixel size is chosen to be ~ FWHM/3

See e.g. http://healpix.jpl.nasa.gov/index.shtml



- HEALPix
  - HEALPix is an acronym for Hierarchical Equal Area isoLatitude Pixelization of a sphere.
  - Subdivision of a spherical surface
  - Each pixel covers the same surface area as every other pixel.
  - The figure below shows the partitioning of a sphere at progressively higher resolutions, from left to right.
  - The green sphere represents the lowest resolution possible: base partitioning of the sphere surface into 12 equal sized pixels. The yellow sphere has a HEALPix grid of 48 pixels, the red sphere has 192 pixels, and the blue sphere has a grid of 768 pixels (~7.3 degree resolution)



  - The pixel centers, represented by the black dots, occur on a discrete number of rings of constant latitude.
  - The number of constant-latitude rings is dependent on the resolution of the HEALPix grid. For the green, yellow, red, and blue spheres shown, there are 3, 7, 15, and 31 constant-latitude rings, respectively.

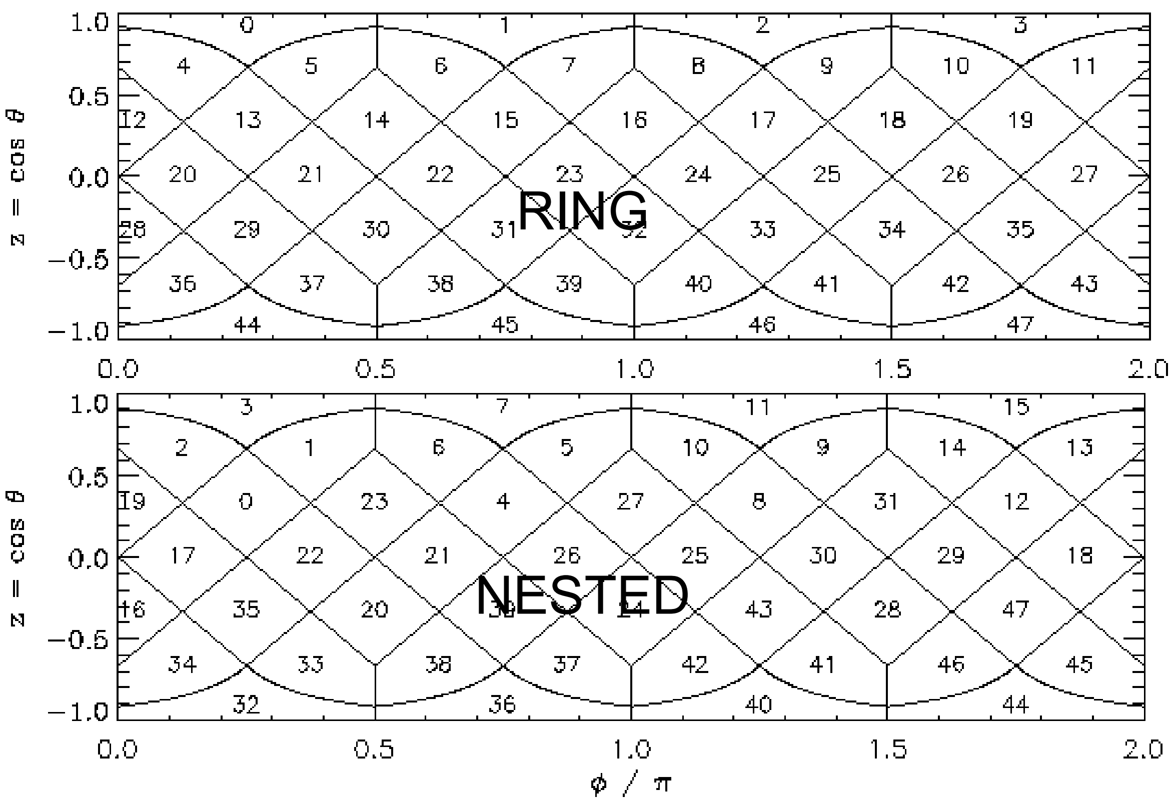
HEALPix

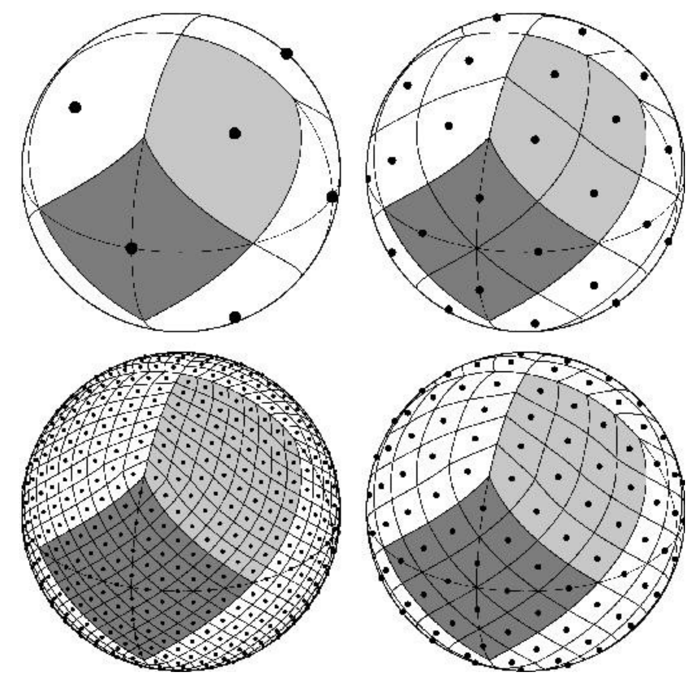
- The principal requirements in the development of HEALPix were to create a mathematical structure which supports a suitable discretization of functions on a sphere at sufficiently high resolution, and to facilitate fast and accurate statistical and astrophysical analysis of massive full-sky data sets.

- HEALPix satisfies these requirements because it possesses the following three essential properties:
  - The sphere is hierarchically tessellated into curvilinear quadrilaterals.
  - Areas of all pixels at a given resolution are identical.
  - Pixels are distributed on lines of constant latitude. This property is essential for all harmonic analysis applications involving spherical harmonics.

HEALPix

- Ogni mappa e' costituita da una serie di parametri che ne definiscono le caratteristiche fondamentali
  - NSIDE = [1,2,4,8,16,32,64,128,256,512,1024,2048,..2^n]
    - definisce il numero di pixel   npix = 12*nside^2
    - Definisce la risoluzione angolare
  - ORDERING = [RING, NESTED]
    - Esistono due diversi ordinamenti [ad anelli, a nido]





HEALPix

- Disponibile a questo link http://healpix.jpl.nasa.gov/

- Fornisce numerosi tools per
  - Leggere e scrivere mappe (fits format)
  - Convertire tra pixel e coordinate
  - Operare sulle mappe (IDL tools)
  - Rappresentare mappe (IDL tools)
  - Trattare polarizzazione con i parametri di Stokes (IDL tools, Fortran tools)
  - Sviluppo in armoniche sferiche e calcolo dello spettro di potenza angolare (Fortran, anafast)
  - Generazione di mappe a partire da armoniche sferiche o da spettri di potenza (Fortran, synfast)
  - ...

What a bolometer measures

Let's consider one single bolometer in the focal plane of a scanning telescope.
In every time sample the bolometer measures a convolution of the sky with the beam in the band.

In a more compact form: $V_i = KB_j + n_i$

Output of sample *i*
Calibration
Brightness of sky pixel *j*
Noise at sample *i*

Calibration includes: spectral behavior of the source, responsivity, efficiency.

The sky is considered pixellized and convolved with the beam.

What a bolometer measures

Let's consider one single bolometer in the focal plane of a scanning telescope.
In every time sample the bolometer measures a convolution of the sky with the beam in the band.

In matrix form:

$$\vec{\mathbf{d}} = K\mathbf{A}\vec{\mathbf{m}} + \vec{\mathbf{n}}$$

Data vector ($\vec{\mathbf{d}}$)	nd elements (very large)
Calibration (K)	
Pointing matrix ($\mathbf{A}$)	(np x nd) elements
Map of the sky ($\vec{\mathbf{m}}$)	np elements (large)
Noise vector ($\vec{\mathbf{n}}$)	nd elements (very large)

The problem is now a large linear system.
Pointing matrix has only one `1` and a lot of `0` per line. It converts from pixel space to time space.

Pointing matrix

$$A_{tp}=\begin{pmatrix}0&1&0\\1&0&0\\0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad nd = 5$$

np = 3

A has (nd x np) elements

At time t=0 the telescope is observing pixel 2
At time t=1 the telescope is observing pixel 1
.....
At time t=4 the telescope is observing pixel 1

A matrix creates a Time Ordered Data from a sky map:

$$\vec{d}=A\vec{m} \Rightarrow \begin{pmatrix}d_{t_0}\\d_{t_1}\\\cdots\\d_{t_4}\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}m_{p_1}\\m_{p_2}\\m_{p_3}\end{pmatrix}=\begin{pmatrix}m_{p_2}\\m_{p_1}\\\cdots\\m_{p_1}\end{pmatrix}$$

nd = 5

A^T sums all the data collected observing the same pixel:

$$A^T\vec{d}=\begin{pmatrix}0&1&0&0&1\\1&0&1&0&0\\0&0&0&1&0\end{pmatrix}\begin{pmatrix}d_{t_0}\\d_{t_1}\\d_{t_2}\\d_{t_3}\\d_{t_4}\end{pmatrix}=\begin{pmatrix}2m_{p_1}\\2m_{p_2}\\m_{p_3}\end{pmatrix}$$

np = 3

A^TA is a diagonal matrix encoding the number of observation falling in the same pixel:

$$A^TA=\begin{pmatrix}0&1&0&0&1\\1&0&1&0&0\\0&0&0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&1&0\\0&0&1\\1&0&0\end{pmatrix}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}$$

np = 3

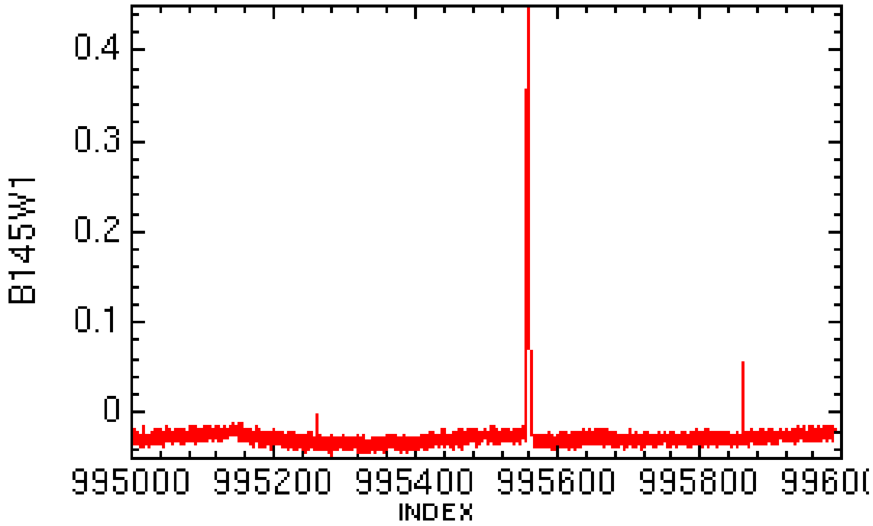
np = 3

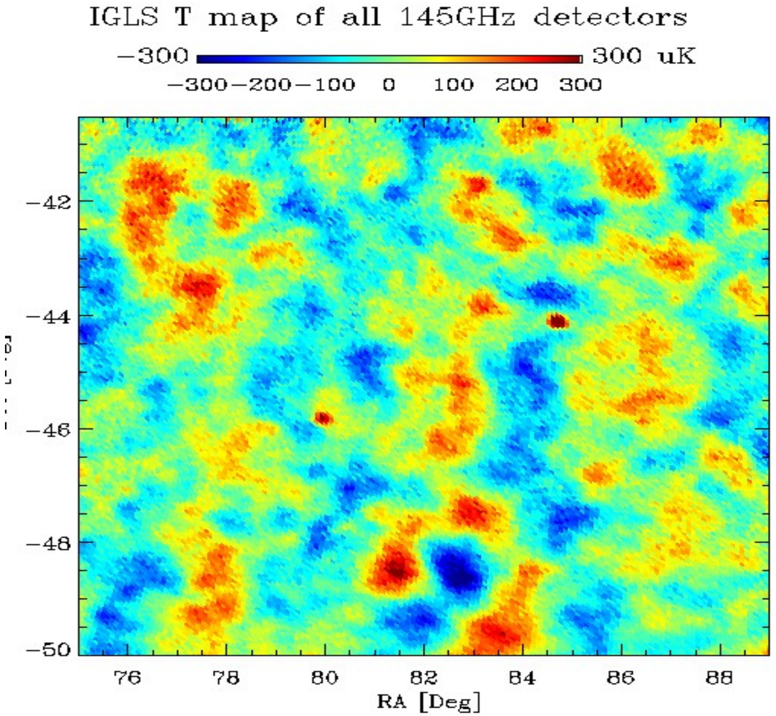
Map-making problem

Time Ordered Data

(pointing)

Maps (T)





Map-making solution, naive solution

Ideal time ordered data: $\vec{d} = A\,\vec{m} + \vec{n}$

A simple solution: $\tilde{m} = (A^T A)^{-1} A^T \vec{d}$

- in every pixel, put the average of all the data falling in that pixel

- this solution is correct if the noise is white (no correlations between observation at different time)

Come fare in pratica, algoritmo naïve

- Supponiamo di avere: ra(t), dec(t), segnale(t), tre vettori della stessa lunghezza, in un file ASCII
- Codice IDL

```
init_healpix                ; inizializza healpix
nside = 1024L               ; definisce la risoluzione
ATd = fltarr(nside^2*12)    ; numeratore
ATA = lonarr(nside^2*12)    ; denominatore (contatore)
map = fltarr(nside^2*12)    ; mappa da costruire

readcol, 'data.dat', ra, dec, signal, count=ndata  ; legge i dati

th = (90-dec)/180*!pi       ; converte in coordinate polari
phi = ra/180*!pi
pix2ang_ring, nside, th, phi, pixel  ; genera vettore pixel

for i=0L, ndata-1 do begin      ; popola numeratore e denominatore
     ATd[pixel[i]] += signal[i]
     ATA[pixel[i]] += 1
endfor

map[*] = !healpix.bad_value           ; valore “bad” per definizione
good = where(ATA gt 0)                ; identifico i pixel osservati
map[good] = Atd[good]/ATA[good]       ; costruisco la mappa nei pixel osservati

write_fits_map, file='mappa.fits', map, nside=nside, coordsys='C', /ring, units='K_cmb'

mollview, 'mappa.fits'      ; disegna la mappa nel file salvato
```

- Metodo di massima verosimiglianza
- Abbiamo dei dati (**d**) e un modello che cerchiamo di ricavare: **m**
- Possiamo definire una verosimiglianza (likelihood):

$$\mathcal{L} = \frac{1}{(2\pi)^{n/2}|\mathbf{N}|^{1/2}} \exp\left[-\frac{1}{2}\left(\mathbf{d}-\mathbf{Am}\right)^T \mathbf{N}^{-1}\left(\mathbf{d}-\mathbf{Am}\right)\right]$$

  - Dove N e' la matrice delle covarianze del rumore

$$\mathbf{N} = \langle(\mathbf{d}-A\mathbf{m})(\mathbf{d}-A\mathbf{m})^T\rangle_{ens}$$

  - Questa likelihood viene usata spesso, quando possiamo ricondurci a variabili distribuite in modo gaussiano (cmb, rumore, no spettro di potenza)

- Abbiamo che un rivelatore misura dati ordinati in tempo

$$\mathbf{d} = \mathbf{A}\mathbf{m} + \mathbf{n}$$

$$\mathbf{n} = \mathbf{d} - \mathbf{A}\mathbf{m}$$

$$\mathbf{N} = \langle \mathbf{n} \cdot \mathbf{n}^T \rangle_{ens}$$

$$\mathcal{L} = \frac{1}{(2\pi)^{n/2}|\mathbf{N}|^{1/2}} \exp\left[-\frac{1}{2}\mathbf{n}^T\mathbf{N}^{-1}\mathbf{n}\right]$$

$$\mathcal{L} = \frac{1}{(2\pi)^{n/2}|\mathbf{N}|^{1/2}} \exp\left[-\frac{1}{2}\left(\mathbf{d} - \mathbf{A}\mathbf{m}\right)^T\mathbf{N}^{-1}\left(\mathbf{d} - \mathbf{A}\mathbf{m}\right)\right]$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{m}} = 0 \qquad \frac{\partial}{\partial \mathbf{m}}\Big(\left(\mathbf{d} - \mathbf{A}\mathbf{m}\right)^T\mathbf{N}^{-1}\left(\mathbf{d} - \mathbf{A}\mathbf{m}\right)\Big) = 0$$

Map-making solution, ideal case

Derivative with **x** and **y** vectors

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

$$\Rightarrow \qquad \frac{\partial}{\partial \mathbf{x}} \mathbf{A}\mathbf{x} = \mathbf{A}$$

$$\frac{\partial}{\partial \mathbf{m}} \left[(\mathbf{d} - \mathbf{A}\mathbf{m})^T \mathbf{N}^{-1} (\mathbf{d} - \mathbf{A}\mathbf{m}) \right] = 0$$

$$-\mathbf{A}^T \mathbf{N}^{-1} (\mathbf{d} - \mathbf{A}\mathbf{m}) - (\mathbf{d} - \mathbf{A}\mathbf{m})^T \mathbf{N}^{-1} \mathbf{A} = 0$$

$$-\mathbf{A}^\mathbf{T} \mathbf{N}^{-1} (\mathbf{d} - \mathbf{A}\mathbf{m}) = 0$$

$$\mathbf{A}^T \mathbf{N}^{-1} \mathbf{d} = \mathbf{A}^T \mathbf{N}^{-1} \mathbf{A}\mathbf{m}$$

$$\tilde{\mathbf{m}} = (\mathbf{A}^T \mathbf{N}^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{N}^{-1} \mathbf{d}$$

Comparison with the naive solution:

$$\tilde{m} = (A^T A)^{-1} A^T \vec{d} \qquad \neq \qquad \tilde{m} = \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} \vec{d}$$

Average of all the observations falling in the same pixel

Optimal noise weighted average of all the observations falling in the same pixel

Map-making solution, ideal case

$$\vec{\mathbf{d}} = \mathbf{A}\vec{\mathbf{m}} + \vec{\mathbf{n}}$$

Ideal time ordered data

$$\tilde{m} = \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} \vec{d}$$

Map solved by inversion regularized by noise

$$\mathbf{N}_{i,j} = \langle n_i n_j^T \rangle \simeq \xi(|t_j - t_i|)$$

time-time noise covariance matrix

Check:

$$\begin{aligned}
\tilde{m} &= \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} \vec{d} \\
&= \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} \left(A\vec{m} + \vec{n}\right) \\
&= \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} A\vec{m} + \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} \vec{n} \\
&= \vec{m} + n_p
\end{aligned}$$

Solved map = sky + pixel noise

Map-making vs naive solutions

If the noise is white, i.e. no correlations between different samples, it can be described using only its RMS amplitude σ, and the noise covariance matrix N can be written as:

$$N=\sigma^2 I \quad \Rightarrow \quad N^{-1}=1/\sigma^2 I$$

$$\begin{aligned}\tilde{m} &= \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} \vec{d} \\ &= \sigma^2 \left(A^T I A\right)^{-1} A^T \sigma^{-2} I \vec{d} \\ &= (A^T A)^{-1} A^T \vec{d}\end{aligned}$$

Which is the naive solution!

In general the white noise assumption is not valid (e.g. 1/f noise), and we have to use the N matrix in the general form:

$$\mathbf{N}_{i,j} = \langle n_i n_j^T \rangle \simeq \xi(|t_j - t_i|)$$

The problem is that both nd (data) and np (pixels) are huge.
For Planck, np = 50 10^6, nd = 25 10^9 per detector
The full noise matrix $N_{i,j}$ cannot be used.
The matrix $\left(A^T N^{-1} A\right)$ is np x np and is hard to invert

Map-making, fast solutions

Natoli et al., astro-ph/0101252

De Gasperis et al, astro-ph/0502142

$$\vec{\mathbf{m}} = \left(\mathbf{A}^{\mathbf{T}}\mathbf{N}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{N}^{-1}\vec{\mathbf{d}}$$

The solution...

$$\left(\mathbf{A}^{\mathbf{T}}\mathbf{N}^{-1}\mathbf{A}\right)\vec{\mathbf{m}} = \mathbf{A}^{\mathbf{T}}\mathbf{N}^{-1}\vec{\mathbf{d}}$$

...can be written like that...

$$Ax = b \Rightarrow Ax - b = 0$$

...as a linear system

$$\varepsilon = \frac{\left\|\left(\mathbf{A}^{\mathbf{T}}\mathbf{N}^{-1}\mathbf{A}\right)\vec{\mathbf{m}} - \mathbf{A}^{\mathbf{T}}\mathbf{N}^{-1}\vec{\mathbf{d}}\right\|}{\left\|\mathbf{A}^{\mathbf{T}}\mathbf{N}^{-1}\vec{\mathbf{d}}\right\|}$$

The algorithm iteratively search the **m** than minimizes this norm using Conjugate Gradient Solver or Jacobi iterative solver

Note that $\frac{d}{dx}\left(\frac{1}{2}Ax^2 - bx\right) = Ax - b$, hence, solving the linear system is the same as finding the minimum of the function $\frac{1}{2}Ax^2 - bx$

Map-making, fast solutions

Natoli et al., astro-ph/0101252

De Gasperis et al, astro-ph/0502142

The tricky part is that under some hypothesis, we don't need to calculate $\mathbf{N}^{-1}$.
If the noise is stationary and the noise correlation is not too long, then **N** is band diagonal and circulant ($n_{i,j} = n_{i+1,j+1} \bmod n_d$).

Then $\mathbf{N}^{-1}\vec{\mathbf{d}} = \mathbf{F}_t^{\dagger}\mathbf{\Xi}^{-1}\mathbf{F}_t\vec{\mathbf{d}}$

where $\mathbf{F_t}$ is the Fourier transform operator and $\mathbf{\Xi}^{-1}$ is a diagonal matrix.

The diagonal part of $\mathbf{\Xi}^{-1}$ is the inverse of the noise power spectrum

Multiplying by $\mathbf{N}^{-1}$ is now a convolution by the PS^{-1}.

Recipe:

the *noise filter* Ξ^{-1} is the Fourier transform of the inverse of the noise power spectrum

Convert the map estimated at the previous iteration in a TOD

$$\left(A^T F_t^{+}\,\Xi^{-1} F_t A\right)\tilde{m} = A^T F_t^{+}\,\Xi^{-1} F_t \vec{d}$$

Same as R.H.S

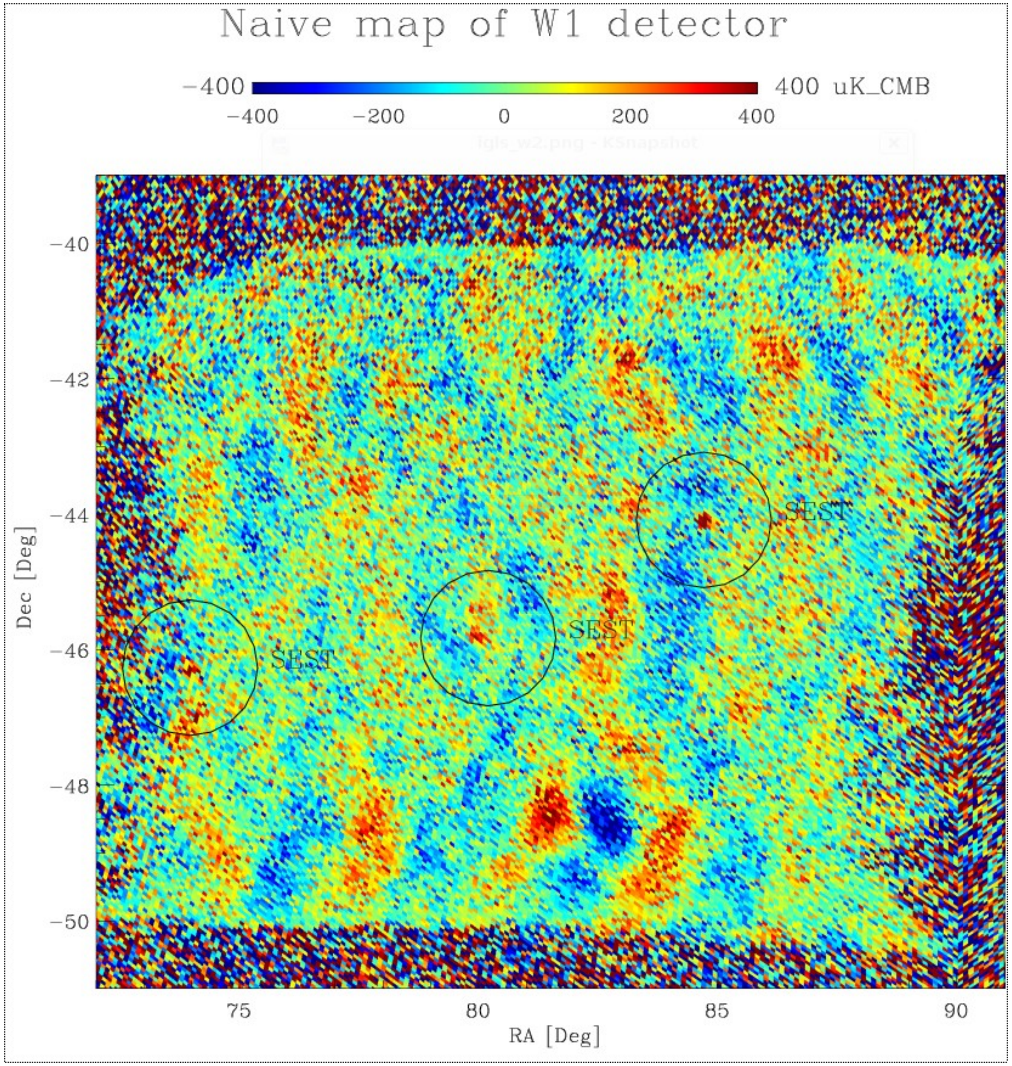
do FFT of the TOD

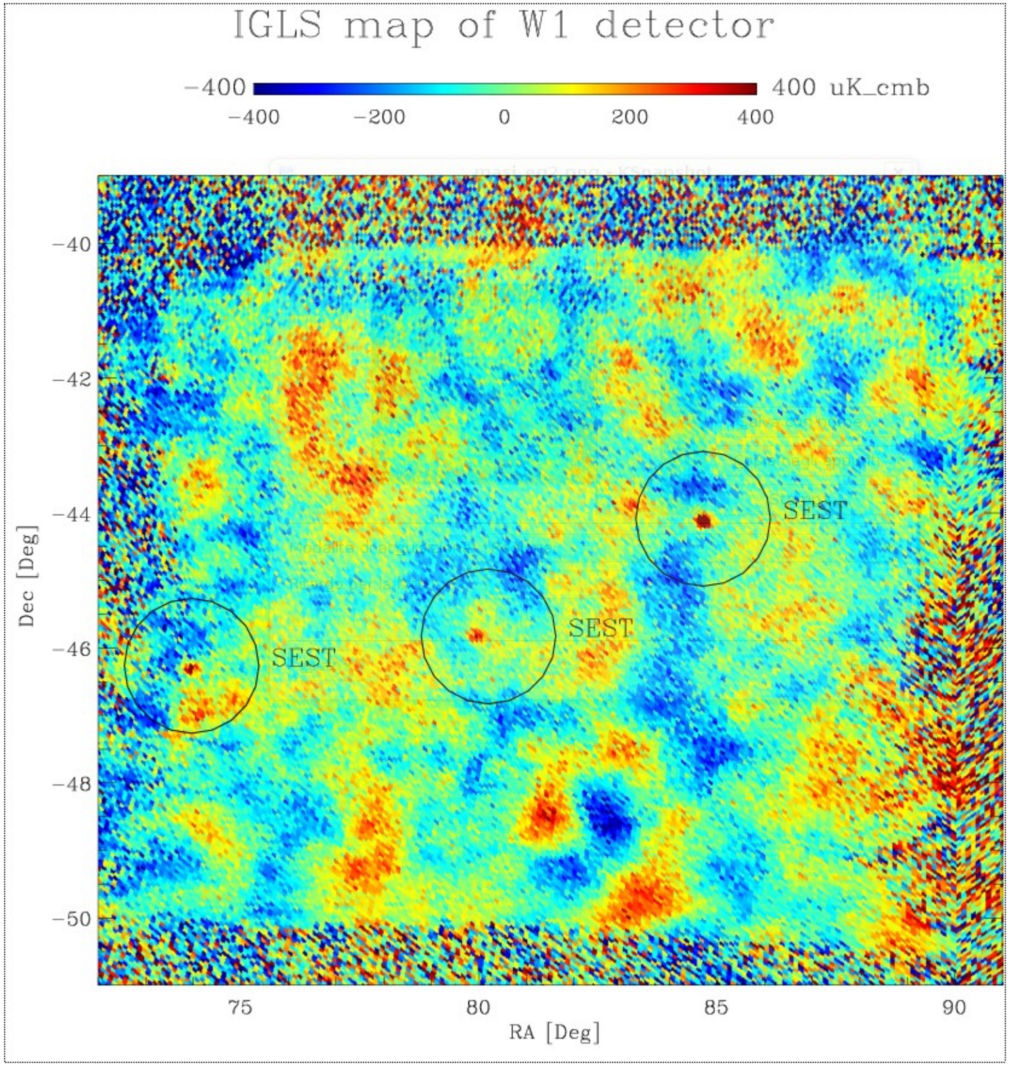
multiply by PS^{-1}

go back to time domain

sum the observation in a map

Map-making: comparison





Being non-optimal, the naive map is more noisy than the IGLS one!

Map-making with several detectors

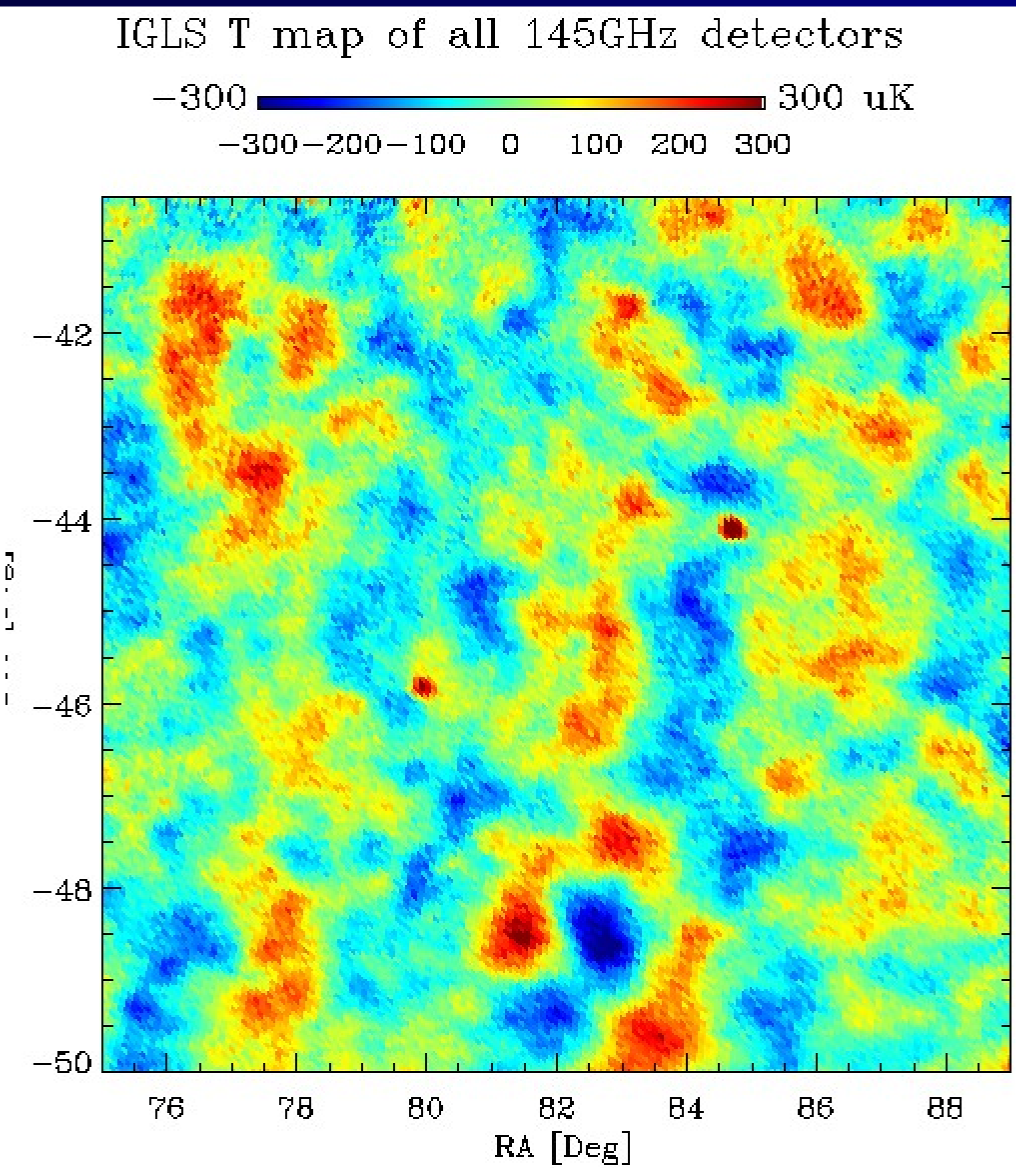
- We can make 1 map for each detector and then do the average, but this is not optimal because the detectors have different noise properties.

- We can build a data vector as:

$$\vec{d} = \left(d_0^1 \ldots d_n^1 \, d_0^2 \ldots d_n^2 \ldots d_0^M \ldots d_n^M \right)$$

and use the optimal map-making, but this is not optimal too

- The solution is to build also a full noise covariance matrix which is block diagonal, each block being associated to one detector. This means we are assuming piece-wise noise stationarity, but the properties of the noise matrix are still valid and we can estimate the map-making solution in exactly the same way as for single detector.



Stokes parameters

- La radiazione incidente incontra un filtro che trasmette ½
  - Filtro 1: I_0 isotropico
  - Filtro 2: I_1 polarizzatore lungo l'asse x
  - Filtro 3: I_2 polarizzatore lungo un asse a 45 gradi
  - Filtro 4: I_3 trasmette solo polarizzazione circolare destra
- Definiamo 4 parametri di Stokes come

$$\begin{aligned} I &= 2I_0 \\ Q &= 2I_1 - 2I_0 \\ U &= 2I_2 - 2I_0 \\ V &= 2I_3 - 2I_0 \end{aligned} \tag{1.35}$$

Stokes parameters

It's easier to explicit this definition using the complex form of the electric fields

$$\begin{cases} E_x(t) = E_{0x}(t)e^{i\delta_x(t)}e^{i(kz-\omega t)} \\ E_y(t) = E_{0y}(t)e^{i\delta_y(t)}e^{i(kz-\omega t)} \\ E_z(t) = 0 \end{cases} \tag{1.36}$$

Then

$$\begin{aligned} I_0 &= \tfrac{1}{2}\langle E_x E_x^* + E_y E_y^* \rangle \\ I_1 &= \langle E_x E_x^* \rangle \\ I_2 &= \langle E_x' E_x'^* \rangle = \langle \tfrac{1}{\sqrt{2}}(E_x + E_y)\tfrac{1}{\sqrt{2}}(E_x^* + E_y^*) \rangle \\ I_3 &= \langle E_x'' E_x''^* \rangle = \langle \tfrac{1}{\sqrt{2}}(E_x + iE_y)\tfrac{1}{\sqrt{2}}(E_x^* - iE_y^*) \rangle \end{aligned} \tag{1.37}$$

where was used $E_x' = \frac{1}{\sqrt{2}}(E_x + E_y)$ (rotation of 45 degrees on the plane $z = 0$ to detect 45 degrees linear polarization) and $E_x'' = \frac{1}{\sqrt{2}}(E_x + iE_y)$: 90 degrees phase-shift of the $\hat{y}$ component and rotation of 45 degrees to measure right circular polarization. Plugging the (1.37) into the (1.35) we find

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* + E_y E_y^* \rangle \\ \langle E_x E_x^* - E_y E_y^* \rangle \\ \langle E_x E_y^* + E_y E_x^* \rangle \\ i\langle E_x E_y^* - E_y E_x^* \rangle \end{bmatrix} = \begin{bmatrix} \langle E_{0x}^2 + E_{0y}^2 \rangle \\ \langle E_{0x}^2 - E_{0y}^2 \rangle \\ \langle 2E_{0x}E_{0y}\cos\delta \rangle \\ \langle 2E_{0x}E_{0y}\sin\delta \rangle \end{bmatrix} \tag{1.38}$$

where $\delta(t) = \delta_x(t) - \delta_y(t)$, $E_{0x}(t)$ and $E_{0y}(t)$ are time dependent and averaging is made on time scales longer than the period T of the wave.

1.5.2 Polarized and unpolarized wave

A monochromatic wave is always polarized. If we define the Stokes parameters for a monochromatic wave, we can remove the time averages in (1.38) and find

$$Q^2 + U^2 + V^2 = I^2 \tag{1.42}$$

A non-monochromatic wave is not necessary polarized. The degree of polarization is defined

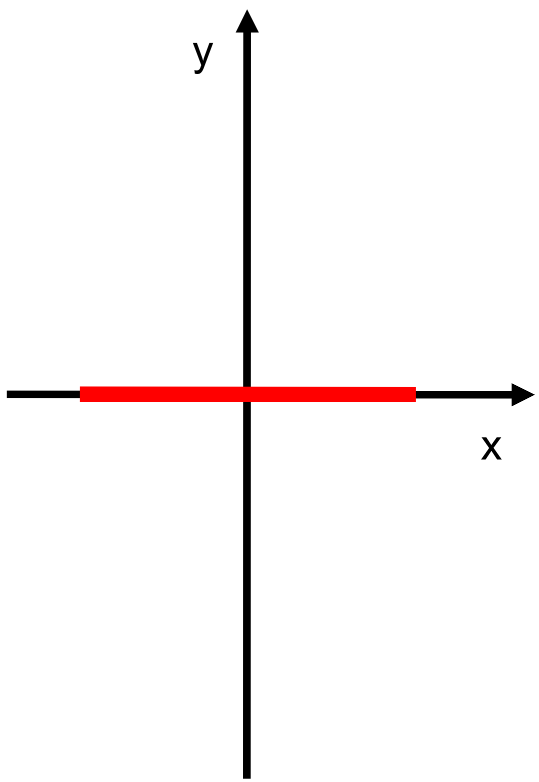
$$\Pi = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \tag{1.43}$$

and is unity for a polarized wave. Linear polarization may be represented by rods

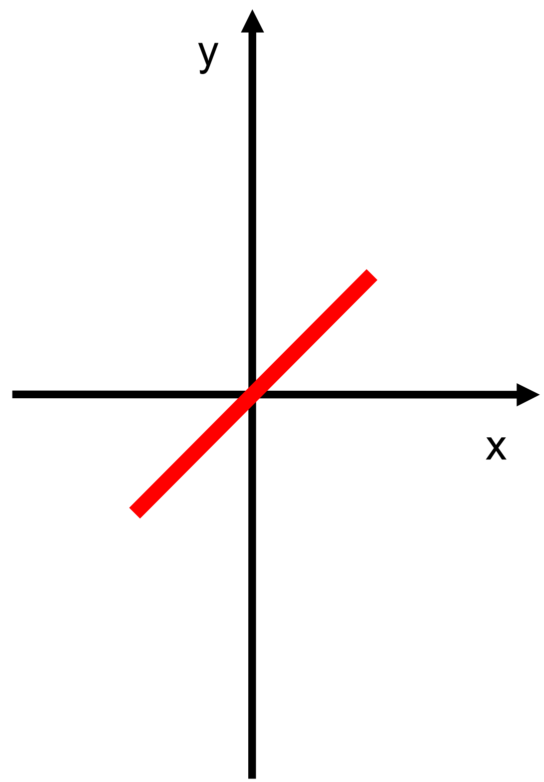
$$\bar{P} = \begin{bmatrix} P\cos\alpha \\ P\sin\alpha \end{bmatrix} \tag{1.44}$$

of magnitude $P = \sqrt{(Q^2 + U^2)}/I$, rotationally invariant, and angle α with respect to the chosen frame: $\alpha = \frac{1}{2}\arctan(U/Q)$. The use of rods instead of vectors reflects the spin-2 nature of linear polarization: 180 degrees rotations return to the original state. To have an unpolarized wave, all the time dependent quantities in the (1.38) δ, E_{0x} and E_{0y} must be random distributions and $\langle E_{0x}(t)\rangle = \langle E_{0y}(t)\rangle$. This is verified, for example, in radiation of thermal origin.

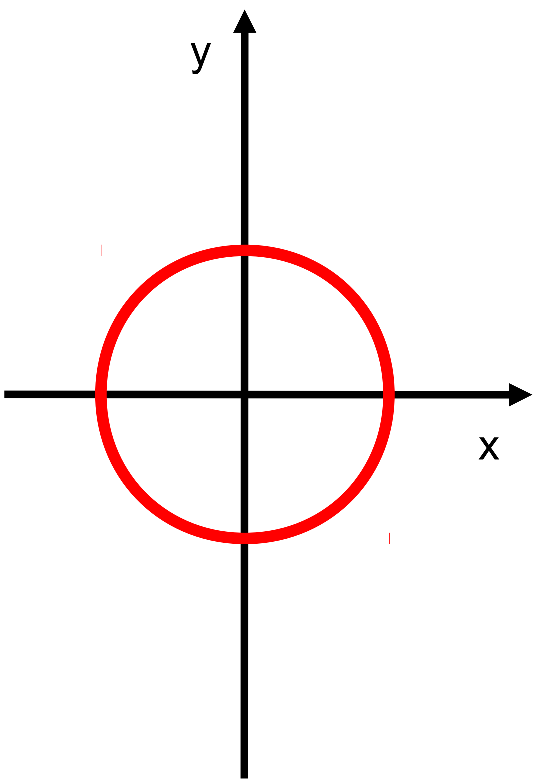
Stokes parameters



$Q = I$
$U = 0$
$V = 0$



$Q = 0$
$U = I$
$V = 0$



$Q = 0$
$U = 0$
$V = I$

Stokes parameters

Rotation of the reference frame by an angle θ around the propagation vector acts on the Stokes parameters as

$$\begin{bmatrix} I' \\ Q' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\theta) & \sin(2\theta) & 0 \\ 0 & -\sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} \qquad (1.41)$$

I and V are invariant, Q and U are said to be spin 2 because they are invariant for 180 degrees rotations.

From equation (1.35) we have

$$I_x = (I + Q)/2$$
$$I_y = (I - Q)/2 \qquad (1.49)$$

A detector with a polarizer along x measures

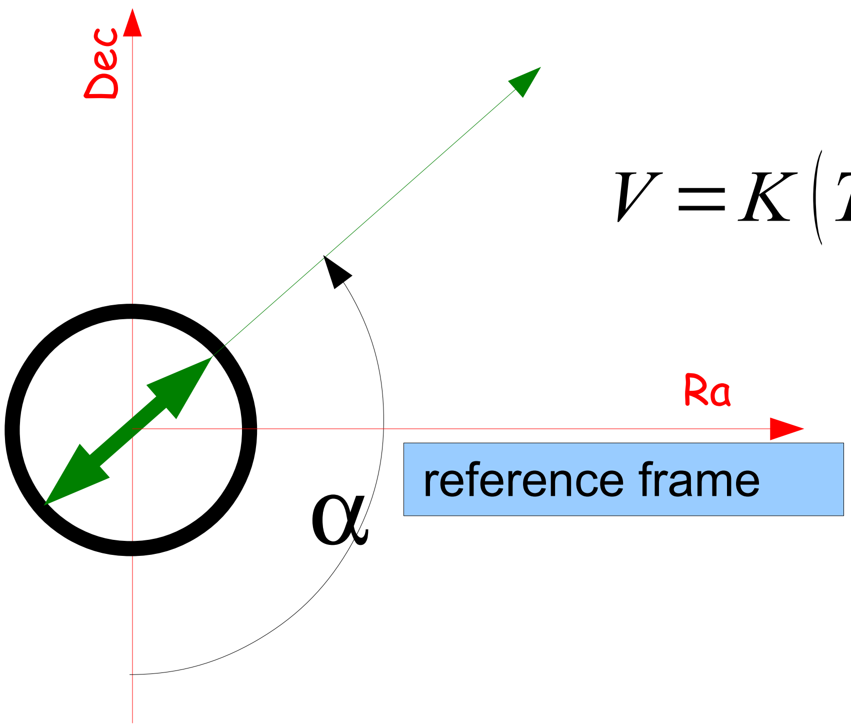
$$I_x = (I + Q)/2$$

A polarization sensitive detector in a generic direction measures

$(T = I)$

$$V = \mathcal{K}\left[T(\hat{n}) + Q(\hat{n})\cos(2\alpha)) + U(\hat{n})\sin(2\alpha)\right] + n \qquad (3.1)$$

With polarization the detector measures



$$V = K\left(T_p + Q_p \cos(2\alpha) + U_p \sin(2\alpha)\right) + n$$

matrix form

$$V = K\begin{bmatrix} 1 & \cos(2\alpha) & \sin(2\alpha) \end{bmatrix} \begin{bmatrix} T_p \\ Q_p \\ U_p \end{bmatrix} + n$$

Stokes parameters Q, U
are defined in a reference frame

Map-making problem

Inverse problem: from sky to TOD

$$
\begin{bmatrix} V_0 \\ V_1 \\ \cdots \\ V_i \\ \cdots \\ V_{Nd} \end{bmatrix} = K \begin{bmatrix} 000100 & 000\cos(2\alpha)00 & 000\sin(2\alpha)00 \\ 010000 & 0\cos(2\alpha)0000 & 0\sin(2\alpha)0000 \\ & \cdots & \\ 001000 & 00\cos(2\alpha)000 & 00\sin(2\alpha)000 \\ & \cdots & \\ 100000 & \cos(2\alpha)00000 & \sin(2\alpha)00000 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ .. \\ T_p \\ \ldots \\ T_{Np} \\ Q_0 \\ Q_1 \\ .. \\ Q_p \\ \ldots \\ Q_{Np} \\ U_0 \\ U_1 \\ .. \\ U_p \\ \ldots \\ U_{Np} \end{bmatrix} + \begin{bmatrix} n_0 \\ n_1 \\ \cdots \\ n_i \\ \cdots \\ n_{Nd} \end{bmatrix}
$$

TOD — pointing matrix (N_d x $3N_p$) — pixellized sky — noise

New pointing matrix, same algebra, same solutions

Naive polarization map-making

- In the polarization mapmaking, the naïve solution is not as simple as average per pixel
- In each pixel there are 3 information to recover (T, Q, U)
- For each pixel, we can write a (nd x 3) system, where nd is the number of data in that pixel

$$d_i = I_p + Q_p \cos(2\alpha_i) + U_p \sin(2\alpha_i)$$

$$\mathrm{A} = \begin{pmatrix} 1 & \cos(2\alpha_1) & \sin(2\alpha_1) \\ 1 & \cos(2\alpha_2) & \sin(2\alpha_2) \\ \ldots & \ldots & \ldots \\ 1 & \cos(2\alpha_{n_d}) & \sin(2\alpha_{n_d}) \end{pmatrix}$$

Naive polarization map-making

- For each pixel

$$\tilde{m} = (A^T A)^{-1} A^T d$$

$$A^T A = \begin{pmatrix} \sum_{n_d} 1 & \sum_{n_d} \cos(2\alpha_i) & \sum_{n_d} \sin(2\alpha_i) \\ \sum_{n_d} \cos(2\alpha_i) & \sum_{n_d} \cos^2(2\alpha_i) & \sum_{n_d} \cos(2\alpha_i)\sin(2\alpha_i) \\ \sum_{n_d} \sin(2\alpha_i) & \sum_{n_d} \cos(2\alpha_i)\sin(2\alpha_i) & \sum_{n_d} \sin^2(2\alpha_i) \end{pmatrix}$$

$$A^T d = \begin{pmatrix} \sum_{n_d} d_i \\ \sum_{n_d} \cos(2\alpha_i) d_i \\ \sum_{n_d} \sin(2\alpha_i) d_i \end{pmatrix}$$

Calibration of the polarimeters

- This is a very simplified statement of the problem

$$V = K\left[T_p + Q_p \cos(2\alpha) + U_p \sin(2\alpha)\right] + n$$

- To be realistic we have to introduce at least the polarimeter efficiency, and the polarimeter orientation angle (in the telescope reference frame)

$$V = K\left[T_p + \rho_{\rm det}\left(Q_p \cos(2(\alpha + \psi_{\rm det}) + U_p \sin(2\alpha + \psi_{\rm det})\right)\right] + n$$

- Moreover, the different detectors must be well matched to be combined, in particular in terms of Optical response (beam)